

Quantum Cohomology of $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and Enumerative Applications

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Abstract

We compute the Small Quantum Cohomology of $\mathbf{H} = \text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ and determine recursively most of the Big Quantum Cohomology. We prove a relationship between the invariants so obtained and the enumerative geometry of hyperelliptic curves in $\mathbb{P}^1 \times \mathbb{P}^1$. This extends the results obtained by Graber [Gr] for $\text{Hilb}^2(\mathbb{P}^2)$ and hyperelliptic curves in \mathbb{P}^2 .

0 Introduction

The Gromov Witten invariants for a smooth projective complex variety X count the (virtual) number of curves on X satisfying some incident conditions. Those related to the genus zero (virtual) curves are equivalent to the Big Quantum Cohomology QH^* of X ; in particular the 3-point invariants give the Small Quantum Cohomology QH_s^* . If X is convex, i.e. $H^1(\mathbb{P}^1, f^*(T_X)) = 0$ for all genus zero stable maps $f : \mathbb{P}^1 \rightarrow X$, the invariants are enumerative since the moduli space is smooth of the expected dimension. In most cases where the invariants have been computed, QH_s^* is explicitly given by generators and relations; other genus zero invariants are recursively determined.

Among the non-convex varieties whose QH^* is known there is the Hilbert scheme $\text{Hilb}^2(\mathbb{P}^2)$, studied by Graber in [Gr]. There he computes QH_s^* explicitly and QH^* recursively using the First Reconstruction Theorem FRT [K-M]. He then relates the invariants to the enumerative geometry of hyperelliptic curves in \mathbb{P}^2 .

In this paper, we study the analogous problem for $Q = \mathbb{P}^1 \times \mathbb{P}^1$. The results we obtain are similar. The main differences are: (1) $H^*(\text{Hilb}^2(Q))$ is not generated by the divisor classes hence a straightforward application of FRT is not enough to determine all the invariants; (2) the group of automorphisms $\text{Aut}(Q)$ has four orbits on $\text{Hilb}^2(Q)$, while $\text{Aut}(\mathbb{P}^2)$ has only two on $\text{Hilb}^2(\mathbb{P}^2)$; (3) we have to be careful about intersection properties of curves, in particular the Position Lemma 1.6 does not apply directly.

In the first section we collect some basic facts about $\mathbf{H} = \text{Hilb}^2(Q)$.

Section two contains a detailed study of the deformation theory of certain stable maps to \mathbf{H} , which is then used to compute some virtual classes and therefore some invariants.

In section three the Small Quantum Cohomology of \mathbf{H} is explicitly computed and an algorithm is given; it determines most of the other genus zero

GW invariants. The methods used are a combination of classical enumerative geometry, application of WDVV and the results of §2.

Section four contains the enumerative applications, i.e. an explicit relation (Theorem 4.12) between genus zero GW invariants and enumerative geometry of hyperelliptic curves on Q , which is completely analogous to that proven for \mathbb{P}^2 by Graber. The proof is however considerably more complicated.

Acknowledgements: I would like to thank Professor Barbara Fantechi for having introduced me to the topic of this paper and for helpful discussions about it.

I am grateful to Professor Angelo Vistoli because during my visit to the Dipartimento di Matematica at the Università di Bologna I could learn a lot from him about the fascinating world of stacks.

A special thank goes to Professor André Hirschowitz who made possible my visit to the Laboratoire J.A. Dieudonné in Nice and to Joachim Kock for helpful discussions at the beginning of my work.

Finally I thank a lot SISSA and ICTP for the stimulating environment these institutions provided me for the last months of my Ph.D.

1 The Hilbert scheme $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

In this section we fix notations and collect some results on the Hilbert scheme $\mathbf{H} := \text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$, whose points parametrize 0-dimensional length-2 closed subschemes Z of $\mathbb{P}^1 \times \mathbb{P}^1$.

Notations and conventions: we work over \mathbb{C} and we identify the variety $\mathbb{P}^1 \times \mathbb{P}^1$ with its image under the Segre embedding, i.e. the smooth quadric Q in \mathbb{P}^3 . We have two rulings on Q , if q_1, q_2 are the two projections on \mathbb{P}^1 , the fibers of q_1 form the first ruling and those of q_2 the second one.

We consider Chow rings with \mathbb{Q} -coefficients. All the varieties in this section have a cellular decomposition, hence we write $A^*(X)$ for the Chow group of X with \mathbb{Q} -coefficients. By [Ful] Example 19.1.11, $A^*(X) \cong H^{2*}(X)$. In particular we can identify them.

Let \mathcal{E} be the sheaf of sections of a vector bundle E , we denote by $\mathbb{P}(E)$ the projective bundle $\text{Proj}(\text{Sym}\mathcal{E})$. Geometrically, points of $\mathbb{P}(E)$ correspond to hyperplanes in the fibers of E .

We indicate a non-reduced 0-dimensional subscheme Z of length 2 of Q as a pair (p, v) where $p \in Q$ is the support of Z and $v \in \mathbb{P}(T_{Q,p})$ is a direction. We call it a non-reduced point of \mathbf{H} .

1.1 Two geometrical descriptions of $\text{Hilb}^2(\mathbb{P}^1 \times \mathbb{P}^1)$

There are two possible geometric descriptions of \mathbf{H} . The first is the standard one as a desingularization of the second symmetric product $\text{Sym}^2(\mathbb{P}^1 \times \mathbb{P}^1)$ (see [Fo]). Let U be the product $Q \times Q$, pr_1, pr_2 the two projections, \tilde{U} the blowup of U along the diagonal $\delta \subseteq U$. The group \mathbb{Z}_2 acts on U fixing δ , so there is an induced action on the blowup \tilde{U} . The Hilbert scheme \mathbf{H} is the quotient scheme \tilde{U}/\mathbb{Z}_2 , hence it is smooth, projective, irreducible and 4-dimensional.

We have the following diagram:

$$\begin{array}{ccccc}
\tilde{\delta} & \xrightarrow{j} & \tilde{U} & \xrightarrow{\theta} & \mathbf{H} \\
bl|_{\tilde{\delta}} \downarrow & & \downarrow bl & & \\
\delta & \xrightarrow{i} & U & \xrightarrow[p_{r_2}]{p_{r_1}} & Q
\end{array}$$

with i, j the natural inclusions, bl the blowup map and θ the quotient map. It induces an isomorphism of \mathbb{Q} -algebras $\theta^* : A^*(\mathbf{H}) \rightarrow A^*(\tilde{U})^{\mathbb{Z}_2}$ which does not respect the degree. We denote by $\Delta \cong \mathbb{P}(T_Q)$ the image in \mathbf{H} of the exceptional divisor $\tilde{\delta}$.

Let h_1, h_2 be the cycle classes of the two rulings on Q . Then $h_0 = [Q]$, $h_1, h_2, h_3 := h_1 h_2$ is a basis of $A^*(Q)$ and $h_r \otimes h_s$, with $0 \leq r, s \leq 3$, a basis of $A^*(U)$. Let ξ be the class of the exceptional divisor $\tilde{\delta}$ in \tilde{U} . Hence as a \mathbb{Q} -algebra $A^*(\mathbf{H})$ is generated by $\xi, T_1 := h_1 \otimes 1 + 1 \otimes h_1, T_2 := h_2 \otimes 1 + 1 \otimes h_2, T_4 := h_3 \otimes 1 + 1 \otimes h_3$.

The Hilbert scheme \mathbf{H} can also be viewed as a blow up of the smooth projective 4-dimensional Grassmannian $\mathbf{G} := \text{Grass}(2, 4)$ of lines in \mathbb{P}^3 . In fact there are two special lines $W_1, W_2 \subseteq \mathbf{G}$ which are disjoint. A point $l_i \in W_i$ represents a line on the i -th ruling of Q , $i = 1, 2$. Denote by W the disjoint union of these special lines, i.e. $W = \{l \in \mathbf{G} : l \subseteq Q\}$. There exists a surjective morphism $\varphi : \mathbf{H} \rightarrow \mathbf{G}$ defined by mapping a point $Z \in \mathbf{H}$ to its associated line l_Z .

Theorem 1.1. *The Hilbert scheme \mathbf{H} is isomorphic to the blow up of the Grassmannian \mathbf{G} along W .*

Proof. The morphism φ is birational, its inverse is a morphism defined by $\varphi^{-1}(r) = r \cap Q$, for all $r \in \mathbf{G} - W$.

If $r \in W$ then the inverse image $\varphi^{-1}(r)$ is $\text{Sym}^2(r) \cong \mathbb{P}^2$, so that $\varphi^{-1}(W)$ is a Cartier divisor in \mathbf{H} . Hence we have a commutative diagram:

$$\begin{array}{ccc}
\mathbf{H} & \xrightarrow{\exists! \alpha} & Bl_W \mathbf{G} \\
& \searrow \varphi & \downarrow \rho \\
& & \mathbf{G}
\end{array}$$

where ρ is the blowup morphism. Since both \mathbf{H} and $Bl_W \mathbf{G}$ are smooth, α is an isomorphism if and only if it is bijective. It is obviously bijective on $\mathbf{G} - W$. By explicit calculations it can be verified that on the exceptional locus the generic fiber of α is a point, i.e. α is a bijection. \square

Let $\sigma_{1,0} \in A^*(\mathbf{G})$ be the Schubert cycle of points $l \in \mathbf{G}$ intersecting a given line $r \subseteq \mathbb{P}^3$. We set $T_3 := \varphi^*(\sigma_{1,0})$.

1.2 The cone of effective curves

We define three 3-codimensional cycle classes and show that they generate the cone of effective curves in \mathbf{H} .

Fix a point $l_1 \in W_1$ and let $C(l_1)$ be a line in the plane $\text{Sym}^2(l_1)$. Note that all the points Z of \mathbf{H} contained in $C(l_1)$ are such that $\text{Supp } Z \subseteq l_1$. We denote by C_1 the corresponding cycle class in $A^3(\mathbf{H})$. We define the class C_2 analogously. Fix a point $p_0 \in Q$ and consider the line $C(p_0) = \mathbb{P}(T_{Q,p_0})$. Let F be the corresponding cycle class in $A^3(\mathbf{H})$. Note that for all $Z \in C(p_0)$ we have $\text{Supp } Z = p_0$. The curves $C(l_1), C(l_2), C(p_0)$ are effective in \mathbf{H} .

Proposition 1.2. *An effective curve in \mathbf{H} is of class $aC_1 + bC_2 + cF$ with $a, b, c \geq 0$.*

Proof. The linear systems associated to T_1, T_2, T_3 (see §1.1) are base-points-free. Since:

$$C_1 \cdot T_2 = 1, C_2 \cdot T_1 = 1, F \cdot T_3 = 1$$

and all other possible intersections give zero, an effective curve in \mathbf{H} is of class $aC_1 + bC_2 + cF$ with $a, b, c \geq 0$. \square

We will write (a, b, c) for the class $aC_1 + bC_2 + cF$.

Remark 1.3. \mathbf{H} is the blowup of \mathbf{G} along W , so by [G-H] p.608 we have $c_1(T_{\mathbf{H}}) = 2(T_1 + T_2)$.

Corollary 1.4. *The expected dimension $ed_{\mathbf{H}}$ of $\overline{M}_{0,n}(\mathbf{H}, (a, b, c))$ is given by the formula:*

$$\exp.\dim \overline{M}_{0,n}(\mathbf{H}, (a, b, c)) = ed_{\mathbf{H}} = 2a + 2b + 1 + n$$

Proof. It follows from 1.3 and the general formula for the expected dimension of a moduli space of stable maps (see [Beh]). \square

1.3 A good \mathbb{Q} -basis for $A^*(\mathbf{H})$

By sections 1.1 and 1.2 we know that $A^*(\mathbf{H})$ can be generated by T_1, T_2, T_3 and T_4 . Note that T_4 can be represented by the cycle class:

$$\Gamma(p) = \{[Z] \in \mathbf{H} : p \in \text{Supp } Z, p \in Q \text{ given point}\}$$

the inverse image of the cycle $\sigma_{2,0}(p)$ in \mathbf{G} via the blowup map φ .

We complete once for all T_1, T_2, T_3, T_4 to a \mathbb{Q} -basis for $A^*(\mathbf{H})$ by adding: $T_0 = [\mathbf{H}]$, $T_5 = T_1T_2$, $T_6 = T_1^2$, $T_7 = T_2^2$, $T_8 = T_1T_3$, $T_9 = T_2T_3$, $T_{10} = C_2 + F$, $T_{11} = C_1 + F$, $T_{12} = C_1 + C_2 + F$ and T_{13} the class of a point.

Remark 1.5. Let ι be the involution of $Q = \mathbb{P}^1 \times \mathbb{P}^1$ defined by $\iota(p, q) = (q, p)$. The induced involution on \mathbf{H} , also denoted by ι , interchanges T_1 and T_2 , T_6 and T_7 , T_8 and T_9 , T_{10} and T_{11} , and leaves the other T_i 's invariant.

1.4 The action of $\text{Aut}(Q)$ on \mathbf{H}

Let $\mathcal{A}_0 = PGL(2) \times PGL(2)$ be the connected component containing the identity in $\mathcal{A} = \text{Aut}(Q)$. Then $\mathcal{A} = \mathcal{A}_0 \sqcup \iota\mathcal{A}_0$. There are four orbits on \mathbf{H} with respect to the \mathcal{A} -action and we can give a description of all of them:

$$\begin{aligned} \Sigma_4 &= \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, l_Z \not\subseteq Q\} \\ \Sigma_3 &= \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, l_Z \subseteq Q\} \\ \Delta_3 &= \{Z \in \mathbf{H} : \text{Supp } Z = p, l_Z \not\subseteq Q\} \\ \Delta_2 &= \{Z \in \mathbf{H} : \text{Supp } Z = p, l_Z \subseteq Q\} \end{aligned}$$

Here indexes are chosen equal to the dimensions of the orbits.

The closed orbit Δ_2 is the disjoint union of two closed subvarieties Δ_2^i , $i = 1, 2$, where $Z \in \Delta_2^i$ if $l_Z \in W_i$.

Note that $\overline{\Delta}_3 = \Delta$ is the divisor of non-reduced points, i.e. the image of $\tilde{\delta}$. The orbit Σ_3 is the disjoint union $\Sigma_3^1 \sqcup \Sigma_3^2$ where

$$\Sigma_3^i = \{Z \in \mathbf{H} : \text{Supp } Z = \{p, q\}, p \neq q, l_Z \in W_i\}$$

In particular the closures $\bar{\Sigma}_3^1, \bar{\Sigma}_3^2$ are the two exceptional divisors \tilde{W}_1, \tilde{W}_2 respectively, of the blowup map $\varphi : \mathbf{H} \rightarrow \mathbf{G}$.

Finally the orbit Σ_4 is open and dense in \mathbf{H} .

\mathbf{H} is an *almost-homogeneous space* since it has a finite number of orbits for the \mathcal{A} -action, hence we can use Graber's Position Lemma.

Lemma 1.6. (Position Lemma-[Gr] Lem.2.5) *Let A be a smooth, almost-homogeneous space under the action of an integral group G , $f : B \rightarrow A$ a morphism with B smooth. Let Γ be a smooth cycle on A which intersects the stratification properly, and Γ_{reg} be the locus in Γ where the intersection with the stratification is transversal. Then:*

1. *for a generic $g \in G$, $f^{-1}(g\Gamma)$ is of pure dimension equal to the expected one;*
2. *the open set (possibly empty) $f^{-1}(g\Gamma_{\text{reg}})$ is smooth.*

Remark 1.7. If in the hypotheses of 1.6 we do not assume B smooth but only pure dimensional we can consider its desingularization $\nu : \tilde{B} \rightarrow B$. Then by applying the Position Lemma to the composition map $\tilde{f} : \tilde{B} \rightarrow A$ we get that $\text{cod}(f^{-1}(g\Gamma) \subseteq \tilde{B})$ is the expected one, i.e. equal to $\text{cod}(g\Gamma \subseteq A)$. Since:

$$\text{cod}(f^{-1}(g\Gamma) \subseteq \tilde{B}) \leq \text{cod}(f^{-1}(g\Gamma) \subseteq B)$$

we have that 1.6-1) holds with the inequality

$$\text{cod}(f^{-1}(g\Gamma) \subseteq B) \geq \text{cod}(g\Gamma \subseteq A)$$

Remark 1.8. Note that \mathcal{A} is not integral, so we will apply the Position Lemma 1.6 to $G = \mathcal{A}_0$.

Remark 1.9. For any $p \in Q$ the cycle $\Gamma(p)$ intersects the stratification properly. In fact $\Gamma(p) \cap \Sigma_4 \cong Q - (l_1(p) \cup l_2(p))$ is obviously a proper intersection and $\Gamma(p) \cap \Delta_2 = \{(p, T_{l_1(p), p}), (p, T_{l_2(p), p})\}$ is 0-dimensional. Since these intersections are non-empty, it is also satisfied $\Gamma(p) \not\subseteq \Sigma_3 \sqcup \Delta_3$.

We set $\Gamma(p)_{\text{reg}}$ to be the locus of $\Gamma(p)$ where the intersection with the stratification is transversal.

Lemma 1.10. *Given a point $p \in Q$, $\Gamma(p)_{\text{reg}}$ is the open subset of $\Gamma(p)$ of points with reduced support.*

Proof. We first prove that $\Delta_2^k \cap \Gamma(p)$, $k = 1, 2$, is not transversal. Δ_2^k is the pullback of the diagonal Δ via the inclusion map $j_k : \tilde{W}_k \hookrightarrow \mathbf{H}$, hence it is a divisor in \tilde{W}_k . By the projection formula we obtain $\Delta_2^k = 4l_k - 2\xi_k$. It is easy to verify that $\Gamma(p)$ intersects Δ_2^k only in one point, but $T_4 \cdot (j_k)_* \Delta_2^k = 2$, this means the intersection is not transversal.

Now consider the closed immersion $f : Q \rightarrow Q \times Q$, defined by $f(q) = (p, q)$. Let θ be the quotient map defined in §1.1. There exists a unique induced closed immersion $\tilde{f} : Bl_p Q = \theta^{-1}(\Gamma(p)) \rightarrow \tilde{U}$, by [Har], Chap.II Cor.7.15. Choosing local coordinates on \mathbf{H} and \tilde{U} it is easy to see that for each $(p, v) \in \theta^{-1}(\Delta)$ the image $d\theta_p(T_{(p,v)}\tilde{U})$ is contained into $T_{\theta(p,v)}\Delta$. As \tilde{f} is a closed immersion and Δ_3 is open dense in Δ , it follows that $\Gamma(p)$ does not intersect Δ_3 transversally.

In order to study the differential of the map $\Gamma(p) \rightarrow \Sigma_3$ it is enough to restrict it to the divisor \tilde{W}_k and to study the differential of $Q - \{p\} \rightarrow Q \times Q - \delta$, $q \mapsto (p, q)$. As $Q = \mathbb{P}^1 \times \mathbb{P}^1$ we can choose affine coordinates on both \mathbb{P}^1 's so that $p = (p_1, p_2)$ and the above map becomes:

$$\begin{array}{ccc} \mathbb{A}^2 - \{(p_1, p_2)\} & \longrightarrow & \mathbb{A}^4 \\ (q_1, q_2) & \mapsto & (p_1, p_2, q_1, q_2) \end{array}$$

Denoting by x_1, x_2, y_1, y_2 the coordinates on \mathbb{A}^4 , the tangent space $T_{(p,q)}\tilde{W}_k$ is the 3-dimensional affine space defined by the equation $x_k - y_k = 0$. In these coordinates $\Gamma(p) - \Delta$ is the set $\{(x_1, x_2, y_1, y_2) : x_1 = p_1, x_2 = p_2\}$ so its tangent space at (p, q) is the 2-dimensional affine space defined by the equations $x_1 = 0, x_2 = 0$. Hence for each $(p, q) \in \Gamma(p) \cap \tilde{W}_k - \Delta$, the space $T_{(p,q)}(\Gamma(p) - \Delta)$ is not contained in $T_{(p,q)}\tilde{W}_k$, that is to say $\Gamma(p)$ intersects Σ_3 transversally. \square

1.5 The locus Δ of non-reduced points of \mathbf{H}

We will refer to the locus Δ of non-reduced points of \mathbf{H} as the *diagonal* of \mathbf{H} . Its class in $A^1(\mathbf{H})$ is $2(T_1 + T_2 - T_3)$. Given a line l in Q , Δ intersects the fiber $\varphi^{-1}(l) \cong \text{Sym}^2(l)$ in a smooth conic.

The natural map $s : \Delta \rightarrow Q$ is defined by mapping a non reduced point to its support so we will call it the *support map*.

Proposition 1.11. *Let $i : \Delta \rightarrow \mathbf{H}$ be the inclusion. Then:*

$$\text{Pic}(\Delta) = \langle \frac{1}{2}T_1, \frac{1}{2}T_2, T_3 \rangle$$

Proof. Let $T_j = i^*T_j$, by abuse of notation. The thesis follows from the equality $\text{Pic}(\mathbb{P}(T_Q)) = \langle s^*\text{Pic}(Q), \mathcal{O}(1) \rangle$. \square

Proposition 1.12. *The effective curves in \mathbf{H} which are contained into Δ are of class (a, b, c) with $a, b, c \geq 0$ and a, b even.*

Proof. Let $C \subseteq \Delta$ be an effective curve of class (α, β, γ) , then i_*C is an effective curve in \mathbf{H} of class (a, b, c) for some non negative integers a, b, c . By the projection formula, $\deg_{\Delta} \frac{1}{2}T_1 \cdot C = \frac{\alpha}{2}$ is an integer number equal to α , hence a is even. The same is true for b , by symmetry. \square

Remark 1.13. By the adjunction formula and 1.3 we get $c_1(T_{\Delta}) = 2T_3$.

1.6 The divisor Σ

Let Σ be given by the disjoint union $\tilde{W}_1 \sqcup \tilde{W}_2$ of the two exceptional divisors of the blowup map φ ; as an element of $A^1(\mathbf{H})$ it is the class $2T_3 - T_1 - T_2$. Note that \tilde{W}_1 is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ because it is the relative Hilbert scheme $\text{Hilb}^2(Q/\mathbb{P}^1)$. Let $\pi_i : \tilde{W}_1 \rightarrow \mathbb{P}^i$ be the natural projection. Then $\text{Pic}(\tilde{W}_1)$ is generated by $L_1 = \pi_1^*\mathcal{O}(1)$, $L_2 = \pi_2^*\mathcal{O}(1)$ and $A_1^{eff}(\tilde{W}_1)$ is generated by $A_1 = [\mathbb{P}^1 \times pt]$ and $A_2 = [pt \times \mathbb{P}^1]$.

Lemma 1.14. *Let $j_1 : \tilde{W}_1 \rightarrow \mathbf{H}$ be the inclusion. Then $j_1^*T_1 = 2L_2$, $j_1^*T_2 = L_1$, $j_1^*T_3 = 2L_2$.*

Proof. It is enough to compute $\deg T_i|_{A_j}$. Fix $l'_2, l''_2 \in W_2$ and consider the curve $C = \{Z : \text{Supp } Z = \{p, q\}, \exists l_1 \in W_1 \text{ with } p = l_1 \cap l'_2, q = l_1 \cap l''_2\}$. Then $A_2 = [C]$. It is easy to see that $j_{1*}(A_1) = C_1$. Hence it is elementary to verify the claim. \square

Corollary 1.15. $c_1(T_{\Sigma}) = 3(T_1 + T_2) - 2T_3$.

Proposition 1.16. *An effective curve in \mathbf{H} which is contained into \tilde{W}_1 is of class (a, b, c) with $a, b, c \geq 0$ and $b = c$ even.*

Proof. We know that $j_{1*}(A_1) = C_1$, hence it is enough to prove that $j_{1*}(A_2) = 2C_2 + 2F$. This follows by Lemma 1.14 and the projection formula. \square

1.7 Description of some effective curves

We describe all the effective connected curves in some cycle classes in $A_1(\mathbf{H})$. For more details see [P] §1.7. In the following sections we will make explicit calculations on the moduli spaces of stable maps involving such curves.

Notations : If $p \in Q$ is a point we will denote by $l_i(p)$ the unique line of the i -ruling on Q going through p .

We will use Propositions 1.12 and 1.16 without explicit reference throughout.

Curves of class $(0, 0, c)$

A curve of class $(0, 0, c)$ is contained in Δ : it is c times a fiber $C(p)$ of the support map over some $p \in Q$.

Curves of class $(1, 0, c), (0, 1, c)$

Since the classes $(1, 0, c), (0, 1, c)$ are symmetric under the involution we can analyse only one of them. We choose $(1, 0, c)$.

Let $\tilde{\varphi} : \mathbf{H} \rightarrow Bl_{W_1} \mathbf{G}$ be the natural map. A curve of class $\beta = (1, 0, 0)$ is a line in a fiber $\tilde{\varphi}^{-1}(l_1) = \text{Sym}^2(l_1)$, for some $l_1 \in W_1$. We will denote it by $C(l_1)$, (see §1.2).

An irreducible curve of class $(1, 0, 1)$ has the form $C(p_1, l_1) = \{Z \in \mathbf{H} : \text{Supp } Z = (p_1, q), q \in l_1\}$, where $p_1 \in Q$ is a fixed point and $l_1 \in W_1$ a fixed line such that $p_1 \notin l_1$. A reducible curve of this class is the union of two irreducible effective components $C(l_1) \cup C(p)$ for some $l_1 \in W_1$ and $p \in l_1$, with $p \in C(l_1) \cap \Delta|_{\varphi^{-1}(l_1)}$. Note that it is contained into $\Delta \cup \Sigma$.

For $c \geq 2$ there are only reducible curves of class $(1, 0, c)$: they are entirely contained into $\Delta \cup \Sigma$ with support $C(l_1) \cup C(p) \cup C(q)$ or $C(l_1) \cup C(p)$ for some $l_1 \in W_1$ and p, q points in $C(l_1) \cap \Delta|_{\varphi^{-1}(l_1)}$.

Curves of class $(1, 1, c), c \leq 1$

Connected curves of class $(1, 1, 0)$ do not exist.

Let C be a reducible curve of type $(1, 1, 1)$. We have three possible decompositions:

- $C(l_1(p)) \cup C(l_2(p)) \cup C(p)$ with p a point of Q ;
- $C(p, l_1) \cup C(l_2(p))$ for a given line l_1 and a given point $p \in Q$, with $C(l_2(p))$ a line in $\text{Hilb}^2(l_2(p))$ passing through (p, q) , $q = l_1 \cap l_2(p)$;
- $C(p, l_2) \cup C(l_1(p))$ symmetrically.

We have two possible families of irreducible curves of class $(1, 1, 1)$. Fix a plane $\Lambda \subseteq \mathbb{P}^3$ and a generic point $q \in \Lambda$, $q \notin Q$. If Λ is generic a curve of such a class is a line $\Lambda(l)$ in $\text{Hilb}^2(\Lambda \cap Q)$, whose points are the closed subschemes Z such that $\text{Supp } Z \subseteq (\Lambda \cap Q)$, $q \in l_Z$. Otherwise the irreducible curve is determined by choosing Λ tangent to Q at a point p and $q \in \Lambda$ such that $q \notin \Lambda \cap Q$. Its points are the closed subschemes Z such that $\text{Supp } Z \cap l_1(p) \neq \emptyset$, $\text{Supp } Z \cap l_2(p) \neq \emptyset$ and $q \in l_Z$. Note that such a curve has only 4 moduli, while the expected dimension is 5.

Remark 1.17. Note that irreducible curves $C(p_1, l_1)$, $C(p_2, l_2)$, $\Lambda(l)$, of class T_{11} , T_{10} and T_{12} respectively, intersect the stratification properly.

2 Virtual Fundamental Classes

This section presents some results about the way of computing some GW invariants we will need in the following. In particular we calculate the virtual fundamental class of two moduli spaces of stable maps on \mathbf{H} .

2.1 Deformation theory on $\overline{M}_{0,n}(\mathbf{H}, \beta)$

We recall the following fundamental result (see [K] Thm.II.1.7):

Theorem 2.1. *If $\mu : C \rightarrow \mathbf{H}$ is a n -pointed stable map and $H^1(C, \mu^*T_{\mathbf{H}}) = 0$, then the forgetful morphism $\eta : \overline{M}_{g,n}(\mathbf{H}, \beta) \rightarrow \mathfrak{M}_{g,n}$ is smooth at the point $[C, x_1, \dots, x_n, \mu]$.*

Remark 2.2. A smooth variety X is called *convex* if $H^1(\mathbb{P}^1, f^*T_X) = 0$ for all genus zero stable maps $f : \mathbb{P}^1 \rightarrow X$. If X is convex then $H^1(C, f^*T_X) = 0$ for all maps $f : C \rightarrow X$, C a genus zero rational curve. Hence the moduli space $\overline{M}_{0,n}(X, \beta)$ is smooth of dimension equal to the expected one, [Al] I.3. Note that every homogeneous variety is convex.

In order to compute some GW invariants, we need to control the smoothness of the moduli space $\overline{M}_{0,n}(\mathbf{H}, \beta)$. \mathbf{H} is not convex but it is an almost-homogeneous space under the action of \mathcal{A} .

Theorem 2.3. *a) Σ is convex hence $\overline{M}_{0,n}(\Sigma, (a, b, c))$ is smooth of the expected dimension $d_{\Sigma} = 3(a + b) - 2c + n$.
b) Δ is convex hence $\overline{M}_{0,n}(\Delta, (a, b, c))$ is smooth of the expected dimension $d_{\Delta} = 2c + n$.*

Proof. Recall that $\Sigma = \tilde{W}_1 \sqcup \tilde{W}_2$. \tilde{W}_k is isomorphic to $\mathbb{P}^2 \times \mathbb{P}^1$ therefore it is homogeneous, hence convex. The formula for d_{Σ} follows from 2.2 and 1.15. This proves statement a).

The support map $s : \Delta \rightarrow Q$ gives the exact sequence:

$$0 \rightarrow T_{\Delta/Q} \rightarrow T_{\Delta} \rightarrow s^*T_Q \rightarrow 0$$

Let $\mu : \mathbb{P}^1 \rightarrow \Delta$ be a stable map, then we have to prove that $H^1(\mathbb{P}^1, \mu^*T_{\Delta/Q})$ vanishes, since Q is homogeneous. The generators of the cone of effective curves in Δ are such that the degree of $T_{\Delta/Q}$ restricted to each of them is non-negative, so $\deg \mu^*T_{\Delta/Q} \geq 0$ and $H^1(\mathbb{P}^1, \mu^*T_{\Delta/Q}) = 0$. As before the formula for d_{Δ} follows from 2.2 and 1.13. This concludes the proof. \square

Theorem 2.4. *If $\mu : C \rightarrow \mathbf{H}$ is a stable map from a genus 0 curve such that no component of C is mapped entirely into $\Delta \cup \Sigma$, then the moduli space $\overline{M}_{0,0}(\mathbf{H}, \beta)$ is smooth at $[C, \mu]$ of the expected dimension.*

Proof. $\mathbf{H} - (\Delta \cup \Sigma)$ is Σ_4 , the open dense orbit for the action on \mathbf{H} induced by \mathcal{A} . The action on Σ_4 is transitive, so we can say that $T_{\mathbf{H}}$ is generically generated by global sections on \mathbf{H} . Let $\mu : C \rightarrow \mathbf{H}$ be as in the hypothesis, then $\mu^*T_{\mathbf{H}}$ is generically generated by global sections on C . This means that $H^1(C, \mu^*T_{\mathbf{H}}) = 0$ and the moduli space $\overline{M}_{0,0}(\mathbf{H}, \beta)$ is smooth at $[C, \mu]$ of the expected dimension by 2.1. \square

2.2 The moduli space $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$

Here and in the following section we prove some results on the virtual fundamental class of two moduli spaces which we will use later on to make explicit calculations.

The virtual fundamental class is defined on $\overline{M}_{g,n}(X, \beta)$ by using a perfect obstruction theory (in the sense of [B-F], [Beh]) or, equivalently, a tangent-obstruction complex (as in [L-T]).

Proposition 2.5. *On the open locus where the moduli stack is smooth, the virtual fundamental class can be computed by integrating the top Chern class of the obstruction bundle.*

Proof. [B-F], Proposition 5.6. \square

We will always denote the obstruction bundle by \mathcal{E} .

If $u : \mathcal{C} \rightarrow \overline{M}_{g,n}(X, \beta)$ is the universal curve with universal map $f : \mathcal{C} \rightarrow X$, then $\mathcal{E} = \text{Ext}_u^2(f^*\Omega_X \rightarrow \Omega_u, \mathcal{O}_{\mathcal{C}})$, whose fiber over a point $[C, x_i, \mu]$ is $\text{Ext}^2(f^*\Omega_X \rightarrow \Omega_C, \mathcal{O}_C)$. It follows that the obstruction bundle on $\overline{M}_{g,n}(X, \beta)$ is the pullback of the obstruction bundle on $\overline{M}_{g,0}(X, \beta)$.

Proposition 2.6. *If the moduli space $\overline{M}_{g,n}(X, \beta)$ is smooth over the Artin stack $\mathfrak{M}_{g,n}$ then $\mathcal{E} = R^1\pi_*ev^*T_X$ is a relative perfect obstruction theory for $\overline{M}_{g,0}(X, \beta)$, with ev the usual evaluation map and π the flat morphism forgetting the n marked points and stabilizing.*

Proof. [Beh], Proposition 5. \square

Let us consider the moduli space $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$.

For $c \geq 1$, a curve of class $(0, 0, c)$ in \mathbf{H} is represented by a c -sheeted cover of \mathbb{P}^1 and it is contained into Δ which is convex. Then the moduli space $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$ is smooth of dimension $2c$ bigger than the expected one, $\text{ed}_{\mathbf{H}} = 1$. The obstruction bundle $\mathcal{E} = R^1\pi_*(ev^*T_{\mathbf{H}})$ has rank $2c - 1$ and its stalk at the point $[C, \mu]$ is $H^1(C, \mu^*T_{\mathbf{H}}) = H^1(C, \mu^*\mathcal{O}_{\mathbb{P}^1}(-2))$. By 2.6 the virtual fundamental class is given by the product:

$$[\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))]^{vir} = [\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))] \cdot c_{2c-1}(\mathcal{E})$$

Proposition 2.7. *Let $g : \overline{M}_{0,0}(\mathbf{H}, (0, 0, c)) \rightarrow Q$ be the map defined by $g([C, \mu]) = \text{Supp } \mu(C)$. It holds:*

$$c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}}) \quad (1)$$

where $\tilde{\mathcal{E}}$ is such that:

$$c_{2c-2}(\tilde{\mathcal{E}}|_{g^{-1}(p)}) = \frac{1}{c^3} \quad (2)$$

for any point $p \in Q$.

Proof. Let $\tilde{ev} : \overline{M}_{0,1}(\mathbf{H}, (0, 0, c)) \rightarrow \Delta$ be the evaluation map into Δ such that the composition with the inclusion $\Delta \hookrightarrow \mathbf{H}$ gives the usual evaluation $ev : \overline{M}_{0,1}(\mathbf{H}, (0, 0, c)) \rightarrow \mathbf{H}$. By [L-Q] Lemma 3.2, \mathcal{E} sits in the exact sequence:

$$0 \rightarrow g^*\mathcal{O}_Q(-K_Q) \rightarrow \mathcal{E} \rightarrow R^1\pi_*\tilde{ev}^*(s^*T_Q \otimes \mathcal{O}_{\Delta}(-1)) = \tilde{\mathcal{E}} \rightarrow 0$$

Hence we get:

$$c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}})$$

Note that the inverse image $g^{-1}(p)$, $p \in Q$, is isomorphic to $\overline{M}_{0,0}(\mathbb{P}^1, c)$, with $\mathbb{P}^1 \cong M_2(p)$ the punctual Hilbert scheme of points on Q at p . With respect to the diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(\mathbb{P}^1, c) & \xrightarrow{ev_1} & \mathbb{P}^1 \\ f \downarrow & & \\ \overline{M}_{0,0}(\mathbb{P}^1, c) & & \end{array}$$

the restriction $\tilde{\mathcal{E}}|_{g^{-1}(p)}$ is isomorphic to $R^1 f_* ev_1^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))$ [L-Q] Rmk.3.1. By Theorem 3.2 in [Man]:

$$c_{2c-2}(R^1 f_* ev_1^*(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))) = \frac{1}{c^3}$$

This concludes the proof. \square

2.3 The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$

In this section we want to study the moduli spaces $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$ and compute (part of) their virtual fundamental classes. By 1.4 they have expected dimension equal to 3.

The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$ is smooth of the expected dimension, because $(1, 0, 0)$ is the class of a curve contained into Σ which is convex and $\overline{M}_{0,0}(\Sigma, (1, 0, 0))$ has the same expected dimension (by 2.3).

We recall from section 1.7 that the only irreducible curves of class $(1, 0, c)$ must have $c = 0$ or 1 ; they are all smooth and rational, and if $c = 1$ they are disjoint from Δ . In $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 1))$ the open substack of maps having irreducible domain is also closed. It is easy to see ([P] §2.6) that $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 1))$ is smooth of the expected dimension.

Irreducible curves of class $(0, 0, c)$ necessarily have $c = 1$, they are smooth, rational and contained in Δ . Hence the domain of every stable map in $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$ is reducible for $c \geq 2$. If $\mu : C \rightarrow \mathbf{H}$ is a stable map of class $(1, 0, c)$ with $c \geq 2$, C has a unique component C_0 mapping isomorphically to a curve of class $(1, 0, 0)$. This defines an \mathcal{A}_0 -equivariant morphism:

$$\tau : \overline{M}_{0,0}(\mathbf{H}, (1, 0, c)) \longrightarrow \overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$$

For $c \geq 1$ let $M(c)$ be the smooth closed substack $ev^{-1}(0)$ in $\overline{M}_{0,1}(\mathbb{P}^1, c)$. Let $M(0)$ be a point. Note that $\dim M(c) = 2c - 2$ for $c \geq 1$ and that $M(1)$ is also a point.

Lemma 2.8. *The general fiber of τ , i.e. over a curve intersecting Δ transversally in two points, is isomorphic to:*

$$\coprod_{\substack{c_i \geq 0 \\ c_1 + c_2 = c \geq 2}} M(c_1) \times M(c_2)$$

In particular τ is smooth over the open dense \mathcal{A}_0 -orbit in $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$.

Proof. Let C be a general curve of class $(1, 0, 0)$. Since C is fixed as well as its intersection points with the diagonal, the only moduli comes from the choice of the sheeted covers of the $(0, 0, 1)$ -curves, i.e. curves in $M(c_i)$, $i = 1, 2$ with $c_1 + c_2 = c \geq 2$, such that the marked point mapping to the origin of the $(0, 0, 1)$ -curve is in $C \cap \Delta$. \square

Remark 2.9. The composition of the inclusion:

$$M(c_1) \times M(c_2) \rightarrow \overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$$

with the forgetful map $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c)) \rightarrow \mathfrak{M}_{0,0}$ is smooth on its image which consists of the (smooth) locus of codimension 2 parametrizing curves with two nodes q_1, q_2 if $c_1, c_2 > 0$, and the divisor parametrizing curves with a node in q_1 if $c_2 = 0$ or in q_2 if $c_1 = 0$.

Remark 2.10. A general fiber of τ has expected dimension equal to zero, since $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$ has expected dimension $\text{ed}_{\mathbf{H}} = 3$ and τ is smooth on the open dense orbit.

Its virtual fundamental class $[\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))]^{vir} \cdot \tau^*[C, f]$ is equal to the sum of the virtual fundamental classes of all components. Moreover each of them must have expected dimension equal to zero.

To calculate the virtual fundamental class of a general fiber of τ we need to know the obstruction bundle \mathcal{E} at one of its points. The following lemma gives a description of the space $H^1(D, \mu^*T_{\mathbf{H}})$ which will permit us to express \mathcal{E} as the cokernel of an injection (see 2.12). Fix $[D, \mu]$ a point in such a fiber:

$$\begin{aligned} \mu : D_0 &\xrightarrow{\cong} C(l_1) \\ \mu : D_i &\xrightarrow{c_i:1} C(p_i) \\ \mu(q_i) &= p_i \in C(l_1) \cap C(p_i) \end{aligned}$$

We assume $c_i > 0$, for $i = 1, 2$; the case with a c_i equal to 0 is similar but easier.

Lemma 2.11. *Let \mathcal{L}_i be the invertible sheaf $\mu^*\mathcal{O}_{C(p_i)}(-2)$ of degree $-2c_i$, $i = 1, 2$. Then:*

$$H^1(D, \mu^*T_{\mathbf{H}}) \cong H^1(D_1, \mathcal{L}_1) \oplus H^1(D_2, \mathcal{L}_2)$$

Proof. We consider the exact sequence in cohomology:

$$H^0(T_{q_1} \oplus T_{q_2}) \rightarrow H^1(D, \mu^*T_{\mathbf{H}}) \rightarrow \bigoplus_{i=1,2} H^1(D_i, \mu^*T_{\mathbf{H}}|_{D_i}) \rightarrow 0$$

The support map $s : \Delta \rightarrow Q$ is a \mathbb{P}^1 -bundle, so the usual exact sequence:

$$0 \rightarrow T_{\Delta} \rightarrow T_{\mathbf{H}}|_{\Delta} \rightarrow \mathcal{N}_{\Delta/\mathbf{H}} \rightarrow 0$$

restricted to a fiber l of s gives $\mathcal{N}_{\Delta/\mathbf{H}}|_l = \mathcal{O}_l(-2)$. Hence we get:

$$H^1(D_i, \mu^*T_{\mathbf{H}}|_{D_i}) = H^1(D_i, \mu^*\mathcal{O}_{C(p_i)}(-2))$$

and the above sequence becomes:

$$T_{q_1} \oplus T_{q_2} \rightarrow H^1(D, \mu^*T_{\mathbf{H}}) \xrightarrow{\vartheta} \bigoplus_{i=1,2} H^1(D_i, \mathcal{L}_i) \rightarrow 0$$

Since $H^1(D_i, \mathcal{L}_i)$ has dimension $2c_i - 1$, ϑ is a surjective morphism between two vector spaces of the same dimension, i.e. it is an isomorphism. \square

Proposition 2.12. *Let \mathcal{L}_i be as in 2.11 and $L_i = \mathcal{E}xt^1(\Omega_D, \mathcal{O}_D)_{q_i}$, $i = 1, 2$, be the line bundle corresponding to the deformations resolving the i -th node. Then the obstruction bundle \mathcal{E} fits in the exact sequence:*

$$0 \rightarrow \bigoplus_{i=1,2} L_i \rightarrow \bigoplus_{i=1,2} H^1(D_i, \mathcal{L}_i) \rightarrow \mathcal{E} \rightarrow 0 \quad (3)$$

In particular it has rank $\sum_{i=1}^2 (2c_i - 2)$.

Proof. In general the absolute obstruction theory $\text{Ext}^\bullet(f^*\Omega_X \rightarrow \Omega_C, \mathcal{O}_C)$ for $\overline{M}_{g,n}(X, \beta)$ is induced by the relative obstruction theory $\text{Ext}^\bullet(f^*\Omega_X, \mathcal{O}_C)$ over $\mathfrak{M}_{g,n}$ (see §2.2 for notations).

The moduli space $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$ is not smooth over $\mathfrak{M}_{0,0}$; however it is smooth over the smooth 2-codimensional locus defined by not smoothening the nodes at q_1, q_2 (see Rmk.2.9). The normal space to this locus is $\bigoplus_{i=1,2} \text{Ext}^1(\Omega_D, \mathcal{O}_D)_{q_i}$. (For more details see [P], Propositions 2.2.4-2.6.10). \square

For each i , let \mathcal{E}_{c_i} be the cokernel of the injection $L_i \rightarrow H^1(D_i, \mathcal{L}_i)$. It is a vector bundle of rank $2c_i - 2$ on $M(c_i)$. It is the one we find when we have only one node on D . Since $\mathcal{E}_{c_1} \oplus \mathcal{E}_{c_2}$ and \mathcal{E} fit into the same exact sequence, it holds $c_{\text{top}}(\mathcal{E}) = c_{\text{top}}(\mathcal{E}_{c_1} \oplus \mathcal{E}_{c_2})$.

In [Gr], Graber constructs a variety X by blowing up \mathbb{P}^2 in a point and then blowing up a point on the exceptional divisor. He gets two exceptional divisors meeting in a node. Let A be the (-1)-curve, B the (-2)-curve and $\beta_c = A + cB$. He shows that the moduli space $\overline{M}_{0,0}(X, \beta_c)$ is smooth of expected dimension zero and isomorphic to $M(c)$. Besides its virtual fundamental class can be realized as the top Chern class of a vector bundle $\tilde{\mathcal{E}}_c$ which sits in the same exact sequence defining the bundle \mathcal{E}_c . Then $c_{\text{top}}(\mathcal{E}_c) = c_{\text{top}}(\tilde{\mathcal{E}}_c)$.

Proposition 2.13. (Graber) *For all $c \geq 2$, $c_{\text{top}}(\tilde{\mathcal{E}}_c) = 0$.*

Proof. This is Proposition 3.5 in [Gr]. \square

Remark 2.14. Let $M^*(c)$ be the fiber over $(0, \infty)$ of the evaluation map $ev = (ev_1, ev_2) : \overline{M}_{0,2}(\mathbb{P}^1, c) \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$. Denote by \mathcal{E}_c^* the obstruction bundle of $M^*(c)$. The following diagram is commutative:

$$\begin{array}{ccc} M^*(c) & \xrightarrow{f} & M(c) \\ \downarrow & & \downarrow \\ \overline{M}_{0,2}(\mathbb{P}^1, c) & \xrightarrow{g} & \overline{M}_{0,1}(\mathbb{P}^1, c) \end{array}$$

where g and f forget the point mapping to ∞ . In particular it can be proved that \mathcal{E}_c^* is the pullback bundle $f^*\mathcal{E}_c$ of the obstruction bundle of $M(c)$, so that its top Chern class vanishes for $c \geq 2$, (see [P] Rmk 2.6.12, Lem.2.6.13).

Theorem 2.15. *The virtual fundamental class of a component of a general fiber of τ is given by:*

$$\begin{aligned} [M(c_1) \times M(c_2)]^{\text{vir}} &= [M(c_1) \times M(c_2)] \quad \text{if } 0 \leq c_1, c_2 \leq 1 \\ [M(c_1) \times M(c_2)]^{\text{vir}} &= 0 \quad \text{otherwise} \end{aligned}$$

Proof. If c_1, c_2 are 0 or 1 then $M(c_1) \times M(c_2)$ is smooth of the expected dimension equal to zero and the virtual fundamental class coincide with the usual fundamental class. If c_1 or c_2 is bigger than or equal to 2 then by 2.13 the top Chern class of the obstruction bundle vanishes. \square

2.4 Some vanishing results

We prove some vanishing results for the GW invariants which are related to the particular geometry of the effective curves involved.

For an exhaustive treatment of the invariants and their properties see for instance [K-M].

Note that propositions 2.5 and 2.6 imply:

Theorem 2.16. *Let $\pi : \overline{M}_{g,n}(X, \beta) \rightarrow \overline{M}_{g,0}(X, \beta)$ be the usual map forgetting the markings and $ev = (ev_1, \dots, ev_n)$ be the evaluation map. Let \mathcal{E} be the obstruction sheaf on $\overline{M}_{g,0}(X, \beta)$. Choose cycles $\Gamma_1, \dots, \Gamma_n$ in X representing the cohomology classes $\gamma_1, \dots, \gamma_n$ such that $ev_i^{-1}(\Gamma_i)$ intersect generically transversally. Then if $A = \pi_*(\cap_i ev_i^{-1}(\Gamma_i))$ is a cycle in the smooth locus of $\overline{M}_{g,0}(X, \beta)$:*

$$\langle \gamma_1 \cdot \dots \cdot \gamma_n \rangle_\beta = \int_A c_{top}(\mathcal{E}) \quad (4)$$

Consider the classes $T_4 = [\Gamma(p)]$, $T_5 - T_4$, $\frac{1}{2}T_6$, $\frac{1}{2}T_7$ in $A^2(\mathbf{H})$.

Proposition 2.17. *For γ equal to one of the above classes, $\langle \gamma \rangle_{(0,0,c)} = 0$.*

Proof. Suppose $c = 1$. A curve $(0, 0, 1)$ is incident to the cycle $\Gamma(p)$ if it is the curve of non-reduced subschemes supported on p , i.e. if it is the fiber over p of the support map s :

$$\Delta \cong \overline{M}_{0,1}(\mathbf{H}, (0, 0, 1)) \xrightarrow{s} \overline{M}_{0,0}(\mathbf{H}, (0, 0, 1)) \cong Q$$

Let ev be the evaluation map $\overline{M}_{0,1}(\mathbf{H}, (0, 0, 1)) \rightarrow \mathbf{H}$.

Since s is flat, $s^*(p) = s^{-1}(p)$ and it is of codimension 2 in $\overline{M}_{0,1}(\mathbf{H}, (0, 0, 1))$. As a set $ev^{-1}(\Gamma(p)) = s^{-1}(p)$, so $ev^*(T_4) = \lambda s^*(p)$ has codimension 2.

$$\langle T_4 \rangle_{(0,0,1)} = \int_{[\overline{M}_{0,1}(\mathbf{H}, (0,0,1))]^{vir}} ev^*(T_4) = \lambda \int_{[\overline{M}_{0,1}(\mathbf{H}, (0,0,1))]} s^*[p \cdot c_{top}(\mathcal{E})] = 0$$

where \mathcal{E} is the obstruction bundle on $\overline{M}_{0,0}(\mathbf{H}, (0, 0, 1))$ and it has rank 1, so that $p \cdot c_{top}(\mathcal{E}) = 0$ on $\overline{M}_{0,0}(\mathbf{H}, (0, 0, 1))$. Curves of type $(0, 0, c)$ intersecting $\Gamma(p)$ are multiple covers of $(0, 0, 1)$, so $\langle T_4 \rangle_{(0,0,c)} = 0$.

The cycle class $T_5 - T_4$ can be represented by the set of subschemes whose support is incident to two lines l_1, l_2 with $l_k \in W_k$, $k = 1, 2$. A curve $(0, 0, 1)$ can meet such a cycle only if it is the curve supported on the incident point $l_1 \cap l_2$. The previous argument works and $\langle T_5 - T_4 \rangle_{(0,0,c)} = 0$.

The cycle classes $\frac{1}{2}T_6, \frac{1}{2}T_7$ are represented by the sets of subschemes with support incident to two lines in the same ruling, so a curve $(0, 0, 1)$ can never meet these cycles. This concludes the proof. \square

Lemma 2.18. *For each $c \geq 1$, $\langle T_8 \rangle_{(0,0,c)} = 4/c^2$. Symmetrically, the formula holds also for T_9 .*

Proof. Consider the diagram:

$$\begin{array}{ccc} \overline{M}_{0,1}(\mathbf{H}, (0, 0, c)) & \xrightarrow{\tilde{ev}} & \Delta \xrightarrow{i} \mathbf{H} \\ \pi \downarrow & & \downarrow s \\ \overline{M}_{0,0}(\mathbf{H}, (0, 0, c)) & \xrightarrow{g} & Q \end{array}$$

where $g([C, \mu]) = \text{Supp } \mu(C)$. Set ev to be the composition map $i \circ \tilde{ev}$.

We know that $c_{2c-1}(\mathcal{E}) = -g^*K_Q \cdot c_{2c-2}(\tilde{\mathcal{E}})$, where \mathcal{E} is the obstruction sheaf on $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$ and $\tilde{\mathcal{E}}$ is the sheaf defined in 2.7. So we have to calculate:

$$\langle T_8 \rangle_{(0,0,c)} = \int_{[\overline{M}_{0,1}(\mathbf{H}, (0,0,c))]} ev^*T_8 \cdot \pi^*(g^*(-K_Q) \cdot c_{2c-2}(\tilde{\mathcal{E}}))$$

Note that a point $[C, x, \mu] \in \overline{M}_{0,1}(\mathbf{H}, (0, 0, c))$ is such that the support of $\mu(C) = \mu(x) = Z$ is a point $p \in Q$, because a curve of class $(0, 0, c)$ is a multiple cover of a fiber of s .

The above diagram is commutative, let f be the composition $g \circ \pi = s \circ \tilde{e}v$. Let h_1 be the cycle class of the first ruling on Q and $\zeta = c_1(\mathcal{N}_{\tilde{\delta}|\tilde{v}})$ of degree -1 on a fiber of the blowup map $\tilde{\delta} \rightarrow \delta$. Then it is easy to verify that $i^*T_8 = 2 \cdot s^*h_1 \cdot \zeta$. We have to calculate the degree:

$$\int_{[\overline{M}_{0,1}(\mathbf{H}, (0, 0, c))]} 2f^*(-K_Q \cdot h_1) \cdot \tilde{e}v^*\zeta \cdot \pi^*(c_{2c-2}(\tilde{\mathcal{E}}))$$

Since $-K_Q \cdot h_1 = 2h_3$ where h_3 is the point-class in $A^2(Q)$, we get:

$$f^*(-K_Q \cdot h_1) \cdot \tilde{e}v^*\zeta = 2\tilde{e}v^*(\zeta \cdot s^*h_3)$$

Let $x \in \Delta$ be a point, we denote by M_1 the inverse image $\tilde{e}v^{-1}(x)$ and by M_0 its image $\pi(M_1) = g^{-1}(s(x))$ in $\overline{M}_{0,0}(\mathbf{H}, (0, 0, c))$. The restricted morphism $\tilde{\pi} : M_1 \rightarrow M_0$ has degree c . In particular:

$$\tilde{\pi}_*[M_1] = c[M_0] = c \cdot g^*[s(x)]$$

Since $\zeta \cdot s^*h_3 = [x]$ is the point-class in Δ , by the projection formula and what we said in §2.2, our invariant is:

$$\begin{aligned} \langle T_8 \rangle_{(0,0,c)} &= \int_{[\overline{M}_{0,1}(\mathbf{H}, (0, 0, c))]} 4\tilde{e}v^*(\zeta \cdot s^*h_3) \cdot \pi^*c_{2c-2}(\tilde{\mathcal{E}}) \\ &= \int_{[M_1]} 4\tilde{\pi}^*c_{2c-2}(\tilde{\mathcal{E}}) = \int_{[M_0]} 4c \cdot c_{2c-2}(\tilde{\mathcal{E}}) \\ &= 4c \cdot c_{2c-2}(\tilde{\mathcal{E}}|_{g^{-1}(s(x))}) = \frac{4}{c^2} \end{aligned}$$

□

With notations as in section 2.3, let $M^* \subseteq \overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$, $c \geq 2$, be the closed subset of stable maps $\mu : D \rightarrow \mathbf{H}$ where the domain curve is reducible and $\mu(D_0) = C(l_1)$ is tangent to the conic defined by Δ in $\text{Sym}^2(l_1)$. Consider the following maps:

$$\overline{M}_{0,3}(\mathbf{H}, (1, 0, c)) \xrightarrow{\pi} \overline{M}_{0,0}(\mathbf{H}, (1, 0, c)) \xrightarrow{\tau} \overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$$

The map π forgets the marked points and (eventually) stabilizes the curve. The map τ is the surjective map defined in §2.3. It is easy to see that fibers of τ are 3-codimensional (see also Remark 2.8.3 in [P] for a detailed proof).

Proposition 2.19. *If $c > 2$ then all GW invariants $\langle \gamma_1 \gamma_2 \gamma_3 \rangle_\beta$ for curves of type $(1, 0, c), (0, 1, c)$ vanish.*

Proof. The two cases are symmetric. We consider only $(1, 0, c)$.

We have seen that such a curve is reducible. It has a component of class $(1, 0, 0)$ not contained into Δ and it decomposes as:

$$(1, 0, c) = (0, 0, c_1) + (0, 0, c_2) + (1, 0, 0)$$

with $c_1, c_2 \geq 0$, $c_1 + c_2 = c$.

We are free to choose a basis of $A^*(\mathbf{H})$ such that every cycle class can be

represented by cycles intersecting the stratification properly. It is enough to prove that GW invariants involving such classes vanish. Choose three of them $\gamma_1, \gamma_2, \gamma_3$ satisfying $\sum \text{cod } \gamma_i = 6$. This condition means we are looking at three possible 3-uples of elements whose codimensions, up to a permutation of indexes, are $(1, 1, 4)$, $(1, 2, 3)$, $(2, 2, 2)$. Consider the diagram:

$$\begin{array}{ccc} A & \xrightarrow{\quad} & \gamma_1 \times \gamma_2 \times \gamma_3 \\ \downarrow & & \downarrow \\ \overline{M}_{0,3}(\mathbf{H}, (1, 0, c)) & \xrightarrow{ev} & \mathbf{H}^3 \end{array}$$

where $A = ev^*(\gamma_1 \times \gamma_2 \times \gamma_3)$. By the Position Lemma $\text{cod } A \geq 6$ (see 1.7). Let π be the flat map defined before and $B = \pi(A)$, then $\text{cod } B \geq 3$. If $\text{cod } B > 3$, the GW invariants vanishes for dimensional reasons, so we can assume $\text{cod } B = 3$.

If the class of a map $[f]$ is in B , then all the points in $\tau^{-1}(\tau([f]))$ are in B , because they differ only by the choice of a multiple cover of $(0, 0, 1)$ and this does not affect incidence conditions. The codimension of a fiber of τ is already equal to 3, so B is a union of finitely many components of fibers of τ . With notations as in section 2.3 the set B is:

$$B = \coprod_{\substack{c_1+c_2=c \\ c_i \geq 0}} M(c_1) \times M(c_2)$$

where $M(0)$ is a point. If $c > 2$ then there exists i such that $c_i > 1$. By 2.15:

$$\langle \gamma_1, \gamma_2, \gamma_3 \rangle_{(1,0,c)} = 0$$

□

3 Quantum Cohomology

Quantum Cohomology is a deformation of the cup product of $A^*(\mathbf{H})$ involving the genus zero Gromov-Witten Invariants. In particular the Small Quantum Cohomology ring $QH_s^*(\mathbf{H})$ of \mathbf{H} incorporates only the genus zero 3-point GW invariants in its product.

In this section we give a presentation of $QH_s^*(\mathbf{H})$ and describe a (partial) algorithm computing all the GW invariants on \mathbf{H} .

We do not give explicitly the complete computations of all the invariants we need, for a more detailed treatment we refer to [P], Chap. 3.

Notations: the cup product in $A^*(\mathbf{H})$ will be denoted by $\alpha \cup \beta$. We will use the symbol $\langle T^n \rangle_\beta$ to denote the GW invariant $\underbrace{\langle T \cdot \dots \cdot T \rangle}_n_\beta$.

3.1 The Small Quantum Cohomology Ring

Let $T_0 = 1, T_1, \dots, T_{13}$ be a homogeneous \mathbb{Q} -basis for $A^*(\mathbf{H})$ such that T_1, T_2, T_3 generate $A^1(\mathbf{H})$. We denote by (g_{ij}) the matrix $(\int_{\mathbf{H}} T_i \cup T_j)$ and by (g^{ij}) its inverse. We introduce formal variables $\{y_0, q_1, q_2, q_3, y_4, \dots, y_{13}\}$ which we will abbreviate as q, y . For β an effective class in $A_1(\mathbf{H})$, the following expression defines a power series in the ring $\mathbb{Q}[[q, y]]$:

$$\Gamma_{ijk} = \sum_{n \geq 0} \sum_{\beta \neq 0} \frac{1}{n!} \langle \gamma^n T_i T_j T_k \rangle_\beta \cdot q^\beta \quad (5)$$

where $\gamma = y_4 T_4 + \dots + y_{13} T_{13}$ and $q^\beta = q_1^b \cdot q_2^a \cdot q_3^c$ for $\beta = (a, b, c)$, (see [G-P]). Note that if one of the indexes i, j, k is zero, then the expression vanishes, because of the condition $\beta \neq 0$.

Consider the free $\mathbb{Q}[[q, y]]$ -module $A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q, y]]$ generated by T_0, \dots, T_{13} . The so called **-product* yields a $\mathbb{Q}[[q, y]]$ -algebra structure on it and it is defined by:

$$T_i * T_j = T_i \cup T_j + \sum_{e, f=0}^{13} \Gamma_{ije} g^{ef} T_f$$

It is well known that the **-product* is commutative, associative, with unit T_0 . In particular let $\gamma_1, \dots, \gamma_n$ be cohomology classes on \mathbf{H} , β in $A_1(\mathbf{H})$ the class of an effective curve and A, B sets of indexes. Then the associativity law reads:

$$\begin{aligned} & \sum \langle T_i \cdot T_j \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} g^{ef} \langle T_k \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2} = \\ & = \sum \langle T_i \cdot T_k \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} g^{ef} \langle T_j \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2} \end{aligned} \quad (6)$$

where the sum is over all the possible partitions $A \cup B = [n]$ of n indexes, all possible sums $\beta_1 + \beta_2 = \beta$ with β_i effective and over $e, f = 0, \dots, 13$.

By definition the Big Quantum Cohomology ring of \mathbf{H} is the $\mathbb{Q}[[q, y]]$ -algebra $(A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[[q, y]], *)$.

The Small Quantum Cohomology ring $QH_s^*(\mathbf{H})$ of \mathbf{H} is defined by setting to zero all the formal variables y_i . Moreover $QH_s^*(\mathbf{H})$ is graded by $\deg q_1 = \deg q_2 = 2$, $\deg q_3 = 0$ and $\deg T_j = \text{cod } T_j$. Since q_1, q_2 have positive degree, we have $T_i * T_j \in A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[q_1, q_2][[q_3]]$. Hence the Small Quantum Cohomology ring of \mathbf{H} is:

$$QH_s^*(\mathbf{H}) = (A^*(\mathbf{H}) \otimes_{\mathbb{Q}} \mathbb{Q}[q_1, q_2][[q_3]], *)$$

It is a deformation of $A^*(\mathbf{H})$ in the usual sense, in fact we can recover the Chow ring of \mathbf{H} by setting all the q_i variables equal to zero.

Let $\mathbb{Q}[Z] = \mathbb{Q}[Z_1, \dots, Z_4]$ and let

$$A^*(\mathbf{H}) = \frac{\mathbb{Q}[Z]}{(f_1, \dots, f_s)}$$

be a presentation with arbitrary homogeneous generators f_1, \dots, f_s for the ideal of relations. Finally let $\mathbb{Q}(q, Z) = \mathbb{Q}[q_1, q_2, Z_1, \dots, Z_4][[q_3]]$. The following proposition is a slightly modified version of [F-P] §10 Prop.11.

Proposition 3.1. *Let f'_1, \dots, f'_s be any homogeneous elements in $\mathbb{Q}(q, Z)$ such that:*

- (i) $f'_i(0, 0, 0, Z_1, \dots, Z_4) = f_i(Z_1, \dots, Z_4)$ in $\mathbb{Q}(q, Z)$,
- (ii) $f'_i(q_1, q_2, q_3, Z_1, \dots, Z_4) = 0$ in $QH_s^*(\mathbf{H})$.

Then the canonical map

$$\frac{\mathbb{Q}(q, Z)}{(f'_1, \dots, f'_s)} \xrightarrow{\varphi} QH_s^*(\mathbf{H})$$

is an isomorphism.

Proof. As in [F-P] we can use a Nakayama-type induction. First we observe that given a homogeneous map $\psi : M \rightarrow N$ between two finitely generated $\mathbb{Q}(q, Z)$ -modules such that the induced map:

$$\frac{M/(q_3)}{(q_1, q_2)} \xrightarrow{\psi_{1,2}} \frac{N/(q_3)}{(q_1, q_2)}$$

is surjective, then $\psi_3 : M/(q_3) \rightarrow N/(q_3)$ is surjective, because q_1, q_2 have positive degree. Since the ideal (q_3) is contained into the radical of Jacobson of $\mathbb{Q}(q, Z)$ and $N = \psi(M) + (q_3)N$, by surjectivity of ψ_3 , it follows that ψ is surjective ([A-M] Cor. 2.7). Hence by hypothesis (i) our map φ is surjective. If \tilde{T}_i , $i = 0, \dots, 13$ are homogeneous lifts to $\mathbb{Q}[q_1, q_2][[q_3]]$ of a basis of $A^*(\mathbf{H})$, exactly the same argument of passing to the quotients shows that their images in $\mathbb{Q}(q, Z)/(f'_1, \dots, f'_s)$ generates this $\mathbb{Q}[q_1, q_2][[q_3]]$ -module. But $QH_s^*(\mathbf{H})$ is free over \mathbb{Q} of rank 14, so φ is an isomorphism. \square

3.2 Explicit calculations of some invariants

We calculate all the monomials arising from the $*$ -product of two generators of $A^*(\mathbf{H})$, except $T_4 * T_4$.

We distinguish different cases.

$$\begin{aligned} T_3 * T_3 &= T_3 \cup T_3 + \sum_{c \geq 1} 2 \{2T_5 + T_6 + T_7 - T_8 - T_9\} q_3^c + \\ &+ \sum_{c \geq 0} c^2 \{ \langle T_{13} \rangle_{(1,0,c)} q_2 q_3^c + \langle T_{13} \rangle_{(0,1,c)} q_1 q_3^c \} T_0 \end{aligned}$$

where we use 2.17, 2.18 and 1.5.

If T_i is a divisor class with $i \neq 3$:

$$T_i * T_3 = T_i \cup T_3 + \sum_{c \geq 0} c \langle T_{13} \rangle_{\beta} \cdot T_0 \cdot q^{\beta} \quad \text{with } \beta = (1, 0, c) \text{ or } (0, 1, c)$$

If T_i, T_j are divisor classes with $i, j \neq 3$:

$$T_i * T_j = T_i \cup T_j + \sum_{c \geq 0} \langle T_i T_j T_{13} \rangle_{\beta} \cdot T_0 \cdot q^{\beta} \quad \text{with } \beta = (1, 0, c) \text{ or } (0, 1, c)$$

If T_i is a divisor class with $i \neq 3$:

$$T_i * T_4 = T_i \cup T_4 + \sum_{\substack{\text{cod } T_e=3 \\ c \geq 0}} \langle T_i T_4 T_e \rangle_{\beta} g^{ef} T_f \cdot q^{\beta} \quad \text{with } \beta = (1, 0, c) \text{ or } (0, 1, c)$$

Finally:

$$T_3 * T_4 = T_3 \cup T_4 + \sum_{\substack{\text{cod } T_e=3 \\ c \geq 0}} c \langle T_4 T_e \rangle_{\beta} g^{ef} T_f \cdot q^{\beta} \quad \text{with } \beta = (1, 0, c) \text{ and } (0, 1, c)$$

where we use 2.17 again.

By the vanishing result 2.19, it is enough to calculate $\langle T_{13} \rangle_{\beta}$ and $\langle T_4, \text{cod } 3 \rangle_{\beta}$ with $\beta = (1, 0, c), (0, 1, c)$, $0 \leq c \leq 2$.

Theorem 3.2. *It holds $\langle T_{13} \rangle_{(1,0,1)} = 2$, $\langle T_4 T_{10} \rangle_{(1,0,1)} = \langle T_4 T_{12} \rangle_{(1,0,1)} = 1$. All other invariants of the form $\langle T_{13} \rangle_{(1,0,c)}$ and $\langle T_4, \text{cod } 3 \rangle_{(1,0,c)}$ are zero.*

Proof. We give a detailed proof for the case $\langle T_{13} \rangle_{(1,0,c)}$; the other proofs are similar, for details see [P], §3.4.

If \mathcal{E} is the rank $d_{\mathbf{H}} - 3$ obstruction bundle on $\overline{M}_{0,0}(\mathbf{H}, (1, 0, c))$, by Theorem 2.16 we have to compute:

$$\langle T_{13} \rangle_{(1,0,c)} = \int_{ev^{-1}(Z)} \pi^* c_{d_{\mathbf{H}}-3}(\mathcal{E})$$

where Z is a generic point of \mathbf{H} representing the class T_{13} and π is the map forgetting a point and stabilizing.

If $c = 0$, we know that $\overline{M}_{0,0}(\mathbf{H}, (1, 0, 0))$ is smooth of the expected dimension $d_{\mathbf{H}} = 3$. In particular the top Chern class of \mathcal{E} gives 1. We can choose a representative Z of the class T_{13} such that $l_Z \notin W_1$, then the fiber $ev^{-1}(Z)$ is empty and the GW invariant vanishes.

If $c = 1$, we have to analyse separately what happens on the two components of the moduli space. We can choose $Z \notin \Delta \cup \Sigma$, with $\text{Supp } Z = \{p_0, q_0\}$, so that reducible curves of type $(1, 0, 1)$ give no contribution to the invariant. Let us consider a stable map with image an irreducible curve. It is a smooth point for the moduli space $\overline{M}_{0,1}(\mathbf{H}, (1, 0, 1))$ which is 4-dimensional in it. Denote by \overline{M}^{irr} the irreducible component parametrizing such maps, then $ev(\overline{M}^{irr}) = \mathbf{H}$. The restricted map $ev : \overline{M}^{irr} \rightarrow \mathbf{H}$ has degree two, because an irreducible curve C of class $(1, 0, 1)$ is completely determined by choosing a line $l_1 \in W_1$ and a point $p_1 \notin l_1$ and all its points are reduced. Hence the fiber over Z contains two points: the isomorphism classes $[\mathbb{P}^1, x, \mu]$ where $\mathbb{P}^1 \cong C(p_0, l_1(q_0))$ or $\mathbb{P}^1 \cong C(q_0, l_1(p_0))$. In the first case, μ is defined by $\mu(t) = (p_0, f(t))$, with $f : \mathbb{P}^1 \rightarrow Q$ a parametrization of $l_1(q_0)$ such that $f(x) = q_0$. Similarly for the other map. Then we have a contribution equal to 2 to the GW invariant.

If $c = 2$, all the curves of class $(1, 0, 2)$ are reducible contained into $\Delta \cup \Sigma$, choosing $Z \notin \Delta \cup \Sigma$ the fiber $ev^{-1}(Z)$ is empty and the GW invariant vanishes. \square

We want to apply proposition 3.1 to get a presentation of $QH_s^*(\mathbf{H})$. Hence we need to write down the 17 relations defining $A^*(\mathbf{H})$ using the $*$ -product. We will denote them by f_i^* . Using associativity, we can calculate almost all the GW invariants we need to reach our aim.

For example the identity $(T_1 * T_1) * T_2 = T_1 * (T_1 * T_2)$ gives:

$$2q_1q_3T_2 + \sum_{\substack{\text{cod } T_e=3 \\ c \geq 0}} \langle T_6T_e \rangle_{(1,0,c)} g^{ef} T_f \cdot q_2q_3^c = \sum_{\substack{\text{cod } T_e=3 \\ c \geq 0}} \langle T_5T_e \rangle_{(0,1,c)} g^{ef} T_f \cdot q_1q_3^c$$

By comparing the coefficients of the variables and by 2.19 we find:

$c = 0$	$\langle T_5T_e \rangle_{(0,1,0)} = 0 \quad \langle T_6T_e \rangle_{(1,0,0)} = 0 \quad \text{for all } T_e \in A^3(\mathbf{H})$
$c = 1$	$\begin{aligned} \langle T_5T_{10} \rangle_{(0,1,1)} &= 0 \\ \langle T_5T_{11} \rangle_{(0,1,1)} &= 2 \\ \langle T_5T_{12} \rangle_{(0,1,1)} &= 2 \end{aligned} \quad \langle T_6T_e \rangle_{(1,0,1)} = 0 \quad \text{for all } T_e \in A^3(\mathbf{H})$
$c \geq 2$	$\langle T_5T_e \rangle_{(0,1,c)} = 0 \quad \langle T_6T_e \rangle_{(1,0,c)} = 0 \quad \text{for all } T_e \in A^3(\mathbf{H})$

Among the necessary invariants which can not be computed with this technique there are those already calculated in Lemma 2.18 and Theorem 3.2.

The rest of them can be worked out by hand like in the following examples. For more details see [P] §3.4.

The invariant $\langle T_{11}T_6 \rangle_{(0,1,c)}$

We want to calculate:

$$\langle T_{11}T_6 \rangle_{(0,1,c)} = \int_{[\overline{M}_{0,2}(\mathbf{H}, (0,1,c))]} ev_1^*(C(p_1, l_1)) \cdot ev_2^*(\gamma) \cdot \pi^*(c_{d_{\mathbf{H}}-3}(\mathcal{E}))$$

where $\gamma = \{Z \in \mathbf{H} : \text{Supp } Z \cap l'_1 \neq \emptyset, \text{Supp } Z \cap l''_1 \neq \emptyset\}$ is a cycle representing T_6 , for fixed lines $l'_1, l''_1 \in W_1$, and \mathcal{E} is the obstruction bundle on $\overline{M}_{0,0}(\mathbf{H}, (0,1,c))$. Both representatives of T_6 and T_{11} can be chosen generic.

If $c = 0$ the invariant gives 1, because of the geometry of a curve of class $(0,1,0)$.

If $c = 1$ we do not have any contribution from the irreducible curves by the genericity assumptions. Let C be a reducible curve of class $(0,1,1)$. It has to be a union $C(l_2) \cup C(p)$ for some $p \in Q$ and $l_2 \in W_2$. Since all the points on $C(p_1, l_1)$ and γ are reduced, C can intersect them only along $C(l_2)$. The line $l_2 = l_2(p_1)$ is then determined. The curve $C(l_2)$ is the line in $\text{Sym}^2(l_2(p_1))$ through $(p_1, l_2 \cap l_1)$ and $(l_2 \cap l'_1, l_2 \cap l''_1)$. Moreover there are two possible points for attaching $C(p)$. This gives a contribution 2 to the invariant.

If $c = 2$ we know that each stable map μ has a reducible domain curve D , in particular $\mu(D)$ is a curve of class $C_2 + c_1F + c_2F$ with $c_1 + c_2 = 2$. The F -components are points of $M(c_1)$ and $M(c_2)$ respectively. As before the intersection points with $C(p_1, l_1)$ and γ lie on the C_2 -component which is completely determined. It intersects Δ in at most two points Z_i , with $\text{Supp } Z_i = q_i$. Then by proposition 2.19 there is only a point satisfying all the incident conditions $[D, x_1, x_2, \mu] \in \overline{M}_{0,2}(\mathbf{H}, (0,1,2))$:

$$\begin{aligned} D &= D_0 \cup D_1 \cup D_2 & D_0 &= C(l_2(p_1)) \\ \mu_*[D_0] &= C_2 & \mu_*[D_i] &= [C(q_i)], \quad i = 1, 2 \\ \mu(x_1) &= (p_1, l_2(p_1) \cap l_1) & \mu(x_2) &= (l_2(p_1) \cap l'_1, l_2(p_1) \cap l''_1) \\ y_i &= D_0 \cap D_i, \quad i = 1, 2 & \mu(y_i) &= q_i, \quad i = 1, 2 \end{aligned}$$

It is a reduced point so it counts with multiplicity one.

The invariant $\langle T_{13}, \text{cod } 3 \rangle_{(1,1,1)}$

Choosing generic representatives for the classes $T_{10}, T_{11}, T_{12}, T_{13}$, stable maps from reducible curves of class $(1,1,1)$ give no contribution because the expected dimension of $\overline{M}_{0,2}(\mathbf{H}, (1,1,1))$ is 7 while reducible curves have less moduli. Then we restrict to study what happens on the component $\overline{M}_{0,2}(\mathbf{H}, (1,1,1))^{irr}$ parametrizing maps from irreducible curves of class $(1,1,1)$, which is smooth of the expected dimension. Fix a generic point Z_0 of \mathbf{H} representing T_{13} with $\text{Supp } Z_0 = \{p_0, q_0\}$.

Lemma 3.3. *If $(ev_1, ev_2) : \overline{M}_{0,2}(\mathbf{H}, (1,1,1))^{irr} \rightarrow \mathbf{H} \times \mathbf{H}$ is the evaluation map and $A = \{Z \in \mathbf{H} : l_Z \cap l_{Z_0} \neq \emptyset\}$. Then $[A] = (ev_2)_* ev_1^*[Z_0]$.*

Proof. Let $(C, x_1, x_2, \mu) \in ev_1^{-1}(Z_0)$ with $Z_0 = \mu(x_1)$ and $Z_1 = \mu(x_2)$. The map μ is an isomorphism with the image curve $\Lambda(l)$, which is a line l in $\text{Hilb}^2(\Lambda \cap Q)$ for Λ generic plane in \mathbb{P}^3 . Since both Z_0 and Z_1 are in $\Lambda \cap Q$, $l_{Z_1} \cap l_{Z_0} \neq \emptyset$, because they lie on the same plane, then $ev_2(ev_1^{-1}Z_0) \subseteq A$.

The set $ev_1^{-1}(Z_0)$ is 3-dimensional as well as A , in particular $[A] = T_3$. The map ev_2 has degree 1 over A , in fact given a generic point $Z \in A$, the lines l_{Z_0}, l_Z generate a unique plane Λ . It cuts a section $\Lambda \cap Q$ on Q and there is a unique line $l \subseteq \text{Hilb}^2(\Lambda \cap Q)$ through Z_0, Z . A curve $\Lambda(l)$ with two markings is uniquely determined. Hence the fiber over Z consists of a unique point $[\mathbb{P}^1, x_1, x_2, \mu]$ where $\mu : \mathbb{P}^1 \rightarrow \mathbf{H}$ is an isomorphism with $\Lambda(l)$ such that $\mu^{-1}(Z_0) = x_1, \mu^{-1}(Z) = x_2$. Then $ev_2(ev_1^{-1}(Z_0))$ is 3-dimensional. This proves the lemma. \square

Corollary 3.4. *For all $T_e \in A^3(\mathbf{H})$:*

$$\langle T_{13} T_e \rangle_{(1,1,1)} = \int_{\mathbf{H}} T_3 \cdot T_e$$

Proof. It follows from 3.3 and 2.16. \square

3.3 A presentation of $QH_s^*(\mathbf{H})$

By the results of the previous section, we can now write down the equations $f_i^*, i = 1, \dots, 17$. Note that we do not write the symmetric equations obtained by simply interchanging T_1 and T_2 .

$$T_3 * T_3 - (T_1 + T_2) * T_3 + T_1 * T_2 - \sum_{c \geq 1} 2q_3^c (2T_1 T_2 + T_1^2 + T_2^2 - T_1 T_3 - T_2 T_3) = 0$$

$$T_1 * T_1 * T_1 - q_1(T_3 - T_1) + 2q_1 q_3(2T_1 + T_2) + q_1 q_3^2(T_1 + 2T_2 - T_3) = 0$$

$$T_1 * T_1 * T_2 - 2T_1 * T_4 = 0$$

$$T_1 * T_1 * T_3 - 2T_1 * T_4 - 2q_1 q_3(T_1 + T_3) - 2q_1 q_3^2(T_1 + 2T_2 - T_3) = 0$$

$$T_1 * T_2 * T_3 - 2T_3 * T_4 = 0$$

$$T_4 * T_4 * T_4 - 2q_1 q_2 q_3^2 T_4 = 0$$

$$T_4 * T_4 * T_1 - 2q_1 q_3 T_2 T_4 - q_1 q_2 q_3 T_3 + 2q_1 q_2 q_3^2(2T_1 + T_2) + \\ - q_1 q_2 q_3^3(2T_1 + 2T_2 - T_3) = 0$$

$$T_4 * T_4 * T_3 - 2(q_1 q_3 T_2 T_4 + q_2 q_3 T_1 T_4) - q_1 q_2 q_3 T_3 - 2q_1 q_2 q_3^2(2T_1 + 2T_2 - T_3) + \\ - 3q_1 q_2 q_3^3(2T_1 + 2T_2 - T_3) = 0$$

$$T_1 * T_1 * T_4 - \frac{1}{2} q_1(T_2 T_3 - T_1 T_2) - q_1 q_3(2T_1 T_2 + T_2^2) - \frac{1}{2} q_1 q_3^2(T_1 T_2 + 2T_2^2 - T_2 T_3) + \\ - q_1 q_2 q_3(1 + 2q_3)T_0 = 0$$

$$T_1 * T_2 * T_4 - T_4 * T_4 - q_1 q_3 T_2^2 - q_2 q_3 T_1^2 - q_1 q_2 q_3(1 + 2q_3)T_0 = 0$$

$$T_1 * T_3 * T_4 - T_4 * T_4 - q_1 q_3(T_1 T_2 + T_2^2 + T_2 T_3) - q_2 q_3 T_1^2 - q_1 q_3^2(T_1 T_2 + 2T_2^2 - T_2 T_3) + \\ - q_1 q_2 q_3(1 + 4q_3 + 3q_3^2)T_0 = 0$$

Theorem 3.5. *The Small Quantum Cohomology ring of \mathbf{H} is:*

$$QH_s^*(\mathbf{H}) = \frac{\mathbb{Q}[q_1, q_2, T_1, T_2, T_3, T_4][[q_3]]}{(f_i^*)_{i=1, \dots, 17}}$$

Proof. The equations f_i^* satisfy the hypotheses of 3.1. \square

Remark 3.6. In the ring $QH_s^*(\mathbf{H})$ the identity $T_4^2 = T_{13}$ corresponds to:

$$T_4 * T_4 = T_{13} + 2q_1q_2q_3^2T_0$$

3.4 The First Reconstruction Theorem

All the classes in the fixed basis of $A^*(\mathbf{H})$ can be written as some product of the divisor classes except T_4 . Hence if we restrict to the subalgebra \mathbf{S} of $A^*(\mathbf{H})$ generated by the divisor classes T_1, T_2, T_3 , we can apply the First Reconstruction Theorem (FRT) ([K-M] Theorem 3.1). It says we can compute all the genus zero GW invariants with arguments in \mathbf{S} by knowing few initial values corresponding to the invariants of the form:

$$\int_{[\overline{M}_{0,2}(\mathbf{H},\beta)]^{vir}} ev^*(\gamma_1 \times \gamma_2)$$

with $ev : \overline{M}_{0,2}(\mathbf{H}, \beta) \rightarrow \mathbf{H}^2$ the usual evaluation map and $\gamma_1, \gamma_2 \in \mathbf{S}$. Since $\text{cod } ev^*(\gamma_1 \times \gamma_2)$ has to be equal to $2a + 2b + 3$ and $\text{cod } \gamma_i \leq 4$ for $i = 1, 2$, we find the upper-bound $a + b \leq 2$. We have only the following cases:

β	$(0, 0, c)$	$(1, 0, c)$	$(0, 1, c)$	$(1, 1, c)$	$(2, 0, c)$	$(0, 2, c)$
$(\text{cod } \gamma_1, \text{cod } \gamma_2)$	$(1, 2)$	$(1, 4)$	$(1, 4)$	$(3, 4)$	$(3, 4)$	$(3, 4)$
		$(2, 3)$	$(2, 3)$			

In sections 2.4 and 3.2 we calculated some of these invariants. The left ones are obtained by means of the associativity. Then we know all of them. This implies that we can calculate all the GW invariants on \mathbf{H} without T_4 among the arguments.

3.5 An algorithm for the tree level GW invariants

Theorem 3.7. Assume we know all the 2-point invariants $\langle \gamma_1 \gamma_2 \rangle_\beta$ with $\gamma_1, \gamma_2 \in \mathbf{S}$, and those of the form $\langle T_4^m \rangle_\beta$, $m \geq 1$. Then we can compute recursively all the invariants of type $\langle T_4^m \gamma_1 \cdots \gamma_n \rangle_\beta$, with $\gamma_i \in \mathbf{S}$ such that $4 \geq \deg \gamma_1 \geq \dots \geq \deg \gamma_n \geq 2$.

Proof. We use equation (3.1) and by induction we suppose to know all the invariants:

$$\langle T_4^r \gamma_1 \cdots \gamma_n \rangle_\beta \quad \text{with } r < m$$

$$\langle T_4^r \gamma_1 \cdots \gamma_s \rangle_\beta \quad \text{with } r + s < m + n$$

$$\langle T_4^m \tilde{\gamma}_1 \cdots \tilde{\gamma}_n \rangle_\beta \quad \text{with } \deg \tilde{\gamma}_n < \deg \gamma_n$$

$$\langle T_4^m \gamma_1 \cdots \gamma_n \rangle_{\beta'} \quad \text{with } \beta - \beta' > 0 \text{ effective}$$

If $m = 0$, there is no problem because each γ_i is in \mathbf{S} .

If $m \geq 1$ and $n = 0$, then we know the values by hypothesis.

If $m = 1$ and $n = 1$, then γ_1 lives necessarily in codimension 3 and we have already calculated all the invariants in section 3.2.

If $m = 1$ and $n \geq 2$, we use (3.1):

$$\begin{aligned} & \sum \langle T_i \cdot T_j \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} g^{ef} \langle T_k \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2} = \\ & = \sum \langle T_i \cdot T_k \cdot T_e \cdot \prod_{a \in A} \gamma_a \rangle_{\beta_1} g^{ef} \langle T_j \cdot T_l \cdot T_f \cdot \prod_{b \in B} \gamma_b \rangle_{\beta_2} \end{aligned}$$

By induction, we know all the invariants with $\beta_i \neq 0$, $i = 1, 2$. We look only to the terms with either β_1 or β_2 equal to zero, i.e. on the left-hand side:

$$\underbrace{\langle T_i \cdot T_j \cdot T_k \cup T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_1} + \underbrace{\langle T_i \cup T_j \cdot T_k \cdot T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_2}$$

on the right-hand side:

$$\underbrace{\langle T_i \cdot T_k \cdot T_j \cup T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_3} + \underbrace{\langle T_i \cup T_k \cdot T_j \cdot T_l \cdot \prod_1^n \gamma_s \rangle_\beta}_{I_4}$$

Since $\gamma_i \in \mathbf{S}$, there exists a decomposition $\gamma_n = \alpha \cup \alpha_1$ with $\alpha_1 \in A^1(\mathbf{H})$ and $\deg \alpha = \deg \gamma_n - 1$. We choose:

$$T_i = T_4, \quad T_j = \gamma_1, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = \gamma_2 \cdot \dots \cdot \gamma_{n-1}$$

Then I_1 is the value $\langle T_4 \gamma_1 \cdot \dots \cdot \gamma_n \rangle_\beta$ we want to know (this will always be the case). Up to a scalar (possibly zero) I_2 is $\langle T_4 \cup \gamma_1 \cdot \alpha \cdot R \rangle_\beta$, all its arguments are in \mathbf{S} . Analogously I_4 is proportional to the known invariant $\langle T_4 \cup \alpha \cdot \gamma_1 \cdot R \rangle_\beta$. Finally in $I_3 = \langle T_4 \cdot \alpha \cdot \gamma_1 \cup \alpha_1 \cdot R \rangle_\beta$ the minimal degree decreased by one. Then we can write I_1 as a combination of lower degree terms. After a finite number of steps we can reduce our problem to the previous case with $n = 1$.

If $m \geq 2$ and $n = 1$, then we have three possibilities for $\text{cod } \gamma_1$. If $\text{cod } \gamma_1 = 4$, we can suppose $\gamma_1 = T_{13}$. We choose:

$$\begin{aligned} T_k, T_l &\in A^2(\mathbf{H}) \cap \mathbf{S} \text{ with } T_k \cup T_l = T_{13} \\ T_i &= T_j = T_4 \\ R &= T_4^{m-2} \end{aligned}$$

We obtain that in $I_2 = \langle T_4^{m-2} T_{13} T_k T_l \rangle_\beta$ we have a lower number of T_4 's as well as in I_3 and I_4 , since $T_4 \cup T_k, T_4 \cup T_l$ are in \mathbf{S} . We can reduce the problem to find $\langle T_4 T_{13} \gamma \rangle_\beta$, with $\gamma \in A^2(\mathbf{H}) \cap \mathbf{S}$, i.e. $m = 1$.

If $\text{cod } \gamma_1 = 3$, then we can decompose it as $\gamma_1 = \alpha \cup \alpha_1$, with $\alpha_1 \in A^1(\mathbf{H})$ as above. Fixing:

$$T_i = T_4, \quad T_j = T_4, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = T_4^{m-2}$$

we get I_2 proportional to $\langle T_4^{m-2} T_{13} \alpha \rangle_\beta$, and we know it by induction. The invariant $I_3 = \langle T_4^{m-1} \alpha \cdot T_4 \cup \alpha_1 \rangle_\beta$ has less T_4 -classes and the minimal degree is lower. Finally I_4 is proportional to $\langle T_4^{m-1} T_{13} \rangle_\beta$, then it is known.

If $\text{cod } \gamma_1 = 2$, we use the same trick with:

$$T_i = T_4, \quad T_j = T_4, \quad T_k = \alpha_1, \quad T_l = \alpha_2, \quad R = T_4^{m-2}$$

where α_1, α_2 are two divisors such that $\alpha_1 \cup \alpha_2 = \gamma_1$. Also in this case we can reduce our problem to the case $m = 1$.

If $m \geq 2$ and $n \geq 2$, then we write $\gamma_n = \alpha \cup \alpha_1$, $\alpha_1 \in A^1(\mathbf{H})$ and we choose:

$$T_i = T_4, \quad T_j = \gamma_1, \quad T_k = \alpha, \quad T_l = \alpha_1, \quad R = T_4^{m-1} \gamma_2 \cdot \dots \cdot \gamma_{n-1}$$

Then I_2, I_4 are invariants with less T_4 's and in I_3 the minimal degree is $\deg \alpha = \deg \gamma_n - 1$. By induction we reduce to the case $n = 1$ or $m = 1$. \square

Remark 3.8. For dimensional reasons, the invariant $\langle T_4^m \rangle_\beta$ vanishes unless m is odd. Moreover we know that for $m = 1, 3$ it is zero.

4 Enumerative applications

We use the results on the Small Quantum Cohomology obtained in the previous section to count how many hyperelliptic curves on Q of given genus and bi-degree pass through a fixed number of generic points. Basically we reduce a question in higher genus to a question about rational curves on the Hilbert scheme \mathbf{H} , as in [Gr]. To do this we need a relationship between our hyperelliptic curves and some rational curves on \mathbf{H} .

By hyperelliptic curve we mean a smooth irreducible projective curve with a choice of hyperelliptic involution, i.e. one with rational quotient. This involution is unique if the genus is greater than or equal to 2.

4.1 The moduli space of hyperelliptic curves mapping to Q

We recall Lemma 2.1 from [Gr].

Lemma 4.1. *If $f : C \rightarrow \mathbb{P}^r$ is a morphism from a hyperelliptic curve such that it does not factor through the hyperelliptic map $\pi : C \rightarrow \mathbb{P}^1$ then $H^i(C, f^*\mathcal{O}(1))$ vanishes for all $i > 0$.*

A similar result holds for maps to Q .

Lemma 4.2. *Let $p_i : Q \rightarrow \mathbb{P}^1$ be the two projections and $\mu : C \rightarrow Q$ be a morphism from a hyperelliptic curve such that $\mu_i := p_i \circ \mu : C \rightarrow \mathbb{P}^1$, $i = 1, 2$, does not factor through the hyperelliptic map. Then $H^i(C, \mu^*T_Q) = 0$ for all $i > 0$.*

Proof. Consider the Euler sequence:

$$0 \rightarrow \mathcal{O} \rightarrow \mathcal{O}(1)^{\oplus 2} \rightarrow T_{\mathbb{P}^1} \rightarrow 0$$

Since $T_Q = p_1^*(T_{\mathbb{P}^1}) \oplus p_2^*(T_{\mathbb{P}^1})$, a surjection is defined:

$$H^1(C, \mu^*p_1^*\mathcal{O}^{\oplus 2}(1) \oplus \mu^*p_2^*\mathcal{O}^{\oplus 2}(1)) \rightarrow H^1(C, \mu^*T_Q) \rightarrow 0$$

By hypothesis $H^j(C, \mu^*p_i^*\mathcal{O}^{\oplus 2}(1)) = 0$ for $j > 0$, so $H^j(C, \mu^*T_Q) = 0$. \square

Let $M_{g,0}(Q, (d_1, d_2))$ be the moduli space of maps $\mu : C \rightarrow Q$ from a smooth irreducible projective curve C of genus g such that $\mu_*[C] = (d_1, d_2)$. Let M_g be the moduli space of semistable projective curves of genus g . We denote by H_g the sub-locus parametrizing hyperelliptic curves. If C is hyperelliptic then the cyclic group of order 2 acts on the space of universal deformations \mathcal{U} of C . It can be proved that the fixed locus $V \subseteq \mathcal{U}$ is the universal deformation space of C as a hyperelliptic curve and it is obviously smooth. It follows that $H_g \subseteq M_g$ is a smooth substack. The cartesian diagram:

$$\begin{array}{ccc} \tilde{H}_g(Q, (d_1, d_2)) & \longrightarrow & H_g \\ \downarrow & & \downarrow \\ M_g(Q, (d_1, d_2)) & \longrightarrow & M_g \end{array}$$

defines the space $\tilde{H}_g(Q, (d_1, d_2))$ parametrizing maps $\mu : C \rightarrow Q$ from a hyperelliptic curve C of genus g with $\mu_*[C] = (d_1, d_2)$. We are interested in the open subset $H_g(Q, (d_1, d_2))$ of maps μ such that the composition maps $\mu_i = p_i \circ \mu : C \rightarrow \mathbb{P}^1$, $i = 1, 2$ do not factor through the hyperelliptic map.

Theorem 4.3. *The natural morphism $\nu : H_g(Q, (d_1, d_2)) \rightarrow H_g$ is smooth.*

Proof. It follows from the vanishing result 4.2; for each $\mu : C \rightarrow Q$ in $H_g(Q, (d_1, d_2))$, we have $H^1(C, \mu^*T_Q) = 0$. Then by theorem 2.1, the forgetful morphism $\overline{M}_{g,0}(Q, (d_1, d_2)) \rightarrow \mathfrak{M}_g$ is smooth in $[\mu]$. Since smoothness is a local property, the theorem follows. \square

Corollary 4.4. *$H_g(Q, (d_1, d_2))$ is smooth and irreducible.*

Proof. Smoothness is a direct consequence of the theorem, since both H_g and ν are smooth.

Since H_g is irreducible, it is enough to prove the fibers of ν are irreducible of constant dimension. A fiber $\nu^{-1}(C)$ is the set of all $\mu : C \rightarrow Q$ of bi-degree (d_1, d_2) such that both μ_1, μ_2 do not factor through the hyperelliptic map. They are two morphisms to the projective line, so they correspond to two line bundles on C of degree d_1, d_2 respectively. We get a morphism $f = (f_1, f_2) : \nu^{-1}(C) \rightarrow \text{Pic}^{d_1}(C) \times \text{Pic}^{d_2}(C)$. By Lemma 4.1 $\text{Im}(f_i)$ is a subset of $\{\mathcal{L}_i : \mathcal{L}_i \text{ is spanned, } h^1(\mathcal{L}_i) = 0\}$. Conversely, for $i = 1, 2$, let W_i be the subset of $\text{Pic}^{d_i}(C)$ of sheaves \mathcal{L}_i such that \mathcal{L}_i is spanned, $h^1(\mathcal{L}_i) = 0$ and \mathcal{L}_i is not a multiple of g_2^1 . Then each $\mathcal{L}_i \in W_i$ is in the image $\text{Im}(f_i)$. W_i is open and dense (if not empty), because $\text{Pic}^{d_i}(C)$ is irreducible. Hence $\text{Im}(f_i)$ contains the open subset W_i and therefore it is irreducible (because W_i is). It follows that $\text{Im}(f)$ is irreducible of dimension $2g$. Each fiber $f^{-1}(\mathcal{L}_1, \mathcal{L}_2)$, $\mathcal{L}_i \in W_i$, is a product $V_1 \times V_2$, where V_i is the open set of pairs of global sections (s_i^1, s_i^2) of \mathcal{L}_i without common zeros, modulo scalars. Hence these fibers are irreducible and they have the same dimension equal to $2(d_1 + d_2) - 2g$, because the first cohomology of \mathcal{L}_i vanishes. Therefore $\nu^{-1}(C)$ is irreducible of dimension $2(d_1 + d_2)$. \square

4.2 The basic correspondence

Let $g : \mathbb{P}^1 \rightarrow \mathbf{H}$ be a map in $M_{0,0}(\mathbf{H}, (a, b, c))$ satisfying the following conditions (\dagger) :

- $g(\mathbb{P}^1)$ intersects Δ transversally
- $g(\mathbb{P}^1)$ is not contained in Σ
- $g(\mathbb{P}^1)$ is disjoint from Δ_2

We can associate to g a map $\mu : C \rightarrow Q$ by:

$$\begin{array}{ccccc} C & \xrightarrow{\quad \mu \quad} & \mathcal{U} & \longrightarrow & Q \\ \pi \downarrow & & \downarrow u & & \\ \mathbb{P}^1 & \xrightarrow{\quad g \quad} & \mathbf{H} & & \end{array}$$

2:1

where \mathcal{U} is the universal family. Then C is a smooth hyperelliptic curve of genus $g_C = a + b - c - 1$ and bi-degree (b, a) . The map $\mu : C \rightarrow Q$ satisfies the following conditions (\dagger) :

- C is a smooth hyperelliptic curve
- both μ_i do not factor through π
- both differentials $d\mu_i$ are injective on ramification points of π

Conversely, let $\mu : C \rightarrow Q$ be an element in $H_g(Q, (d_1, d_2))$, with hyperelliptic map $\pi : C \rightarrow \mathbb{P}^1$. If it satisfies (\dagger) , there exists a canonical map $g : \mathbb{P}^1 \rightarrow \mathbf{H}$ which induces it (see [P], §4.2). More precisely, let us consider

the open subset $M_{0,0}^{\text{tr}}(\mathbf{H}, \beta) \subseteq M_{0,0}(\mathbf{H}, \beta)$ of maps from irreducible rational curves fulfilling (\dagger) . Let $H_g^{\text{tr}}(Q, (d_1, d_2)) \subseteq H_g(Q, (d_1, d_2))$ be the open subset parametrizing maps μ satisfying (\ddagger) .

Theorem 4.5. *There is a canonical isomorphism:*

$$H_g^{\text{tr}}(Q, (d_1, d_2)) \cong M_{0,0}^{\text{tr}}(\mathbf{H}, (d_2, d_1, d_1 + d_2 - g - 1))$$

Proof. The proof of Theorem 2.4 in [Gr] never makes use of the fact that the curves are in \mathbb{P}^2 , then it works also for hyperelliptic curves on Q . \square

4.3 Enumerative results

By Theorem 4.5 we might expect a relationship between the number n of hyperelliptic curves on Q of bi-degree (d_1, d_2) and genus g passing through r general points and some Gromov-Witten invariants involving the cycle $\Gamma(p)$, $p \in Q$, and the moduli space $\overline{M}_{0,r}(\mathbf{H}, (d_2, d_1, d_1 + d_2 - g - 1))$. In particular we need to exclude undesired contributions to the number n coming from stable maps either living in the wrong dimension or with a reducible domain curve.

Theorem 4.6. *Fix an effective class $\beta = (a, b, c) \in A_1(\mathbf{H})$, $a + b \geq 1$, and r general points p_1, \dots, p_r on Q with $r = 2a + 2b + 1$. Then:*

1. *there exists at most a finite number of irreducible rational curves of class β incident to all the cycles $\Gamma(p_i)$;*
2. *all such curves intersect $\Delta \cup \Sigma$ in points disjoint from the $\Gamma(p_i)$;*
3. *given any arbitrary stable map $\mu : C \rightarrow \mathbf{H}$ of class β incident to all the cycles $\Gamma(p_i)$, then C has a unique irreducible component which is not entirely mapped into $\Delta \cup \Sigma$, such a component is of class (a, b, c_0) , where $c_0 \leq c$.*

Corollary 4.7. *In the same assumptions, let us consider the usual evaluation map $ev : \overline{M}_{0,r}(\mathbf{H}, (a, b, c)) \rightarrow \mathbf{H}^r$. Then $ev^{-1}(\prod \Gamma(p_i))$ is zero dimensional and smooth.*

Proof. Theorem 4.6 (which is proven in §4.4) says that given a stable map $\mu : C \rightarrow \mathbf{H}$ satisfying all incident conditions, aside from the distinguished component of C of class (a, b, c_0) , all other components are of type $(0, 0, c')$ and they are entirely mapped into Δ . So they are multiple covers of \mathbb{P}^1 . Moreover, adding a component of type $(0, 0, c')$ to a stable map can never cause it to be incident to any extra $\Gamma(q)$, since it would force another component of the curve to meet the corresponding cycle. Finally, different $(0, 0, c')$ -components are disjoint, since they are different fibers of the support map s , hence they must be incident to the distinguished component, C been connected.

We conclude that the source curve looks like a comb, with the component of class (a, b, c_0) as the handle and the components of class $(0, 0, c')$ as the teeth. We get exactly the same picture obtained in [Gr].

There is a finite number of such curves. Infact, if C is irreducible, then Theorem 4.6 confirms our assertion. If C is reducible, we have only a finite number of possibilities for the multiple covers of a $(0, 0, 1)$ -curve and only a finite number of points of intersection of the distinguished component with Δ . So there are only finitely many potential image curves for stable maps incident to all of the cycles.

To prove smoothness, by Position Lemma 1.6-2 it is enough to prove that $ev^{-1}(\Gamma) = ev^{-1}(\Gamma_{\text{reg}})$, where $\Gamma = \prod \Gamma(p_i)$.

By Lemma 1.10 $\Gamma_{reg} = \prod(\Gamma(p_i) - \Delta)$ and Theorem 4.6-2 ensures that $ev^{-1} \cap ev_i^{-1}(\Gamma(p_i) \cap \Delta)$. \square

With notations as in §2.3, let A_1 be the product $M(c_1) \times \dots \times M(c_m)$ with $\sum c_i = c$ and $c_i = 1$ for all i , and let A_2 be the disjoint union of all the other products with $\sum c_i = c$.

Corollary 4.8. *In the same assumptions, we denote by A the image of $ev^{-1}(\prod \Gamma(p_i))$ in $\overline{M}_{0,0}(\mathbf{H}, (a, b, c))$, via the usual map forgetting the markings (and stabilizing). Then $A = A_1 \sqcup A_2$. In particular $\deg A_1 = \langle T_4^r \rangle_{(a,b,c)}$.*

Proof. Theorem 4.6 says that the only moduli in the choice of a stable map meeting all the $\Gamma(p_i)$ comes from the choice of multiple covers of the $(0, 0, 1)$ curve. Then as a set, each component of A decomposes as a product:

$$M(c_1) \times M(c_2) \times \dots \times M(c_m)$$

with $c_1 + \dots + c_m = c$. In particular A is contained in the smooth locus of $\overline{M}_{0,0}(\mathbf{H}, (a, b, c))$. Theorems 2.15 and 2.16 conclude the proof. \square

Definition 4.9. *A curve C in \mathbf{H} of class (a, b, c) is comblike if it is a union of an irreducible (a, b, c_0) -curve, $c_0 \leq c$, and $c - c_0$ disjoint rational curves, mapping isomorphically onto a $(0, 0, 1)$ -curve.*

Remark 4.10. All the stable maps in A_1 have comblike curves as domain.

Definition 4.11. *Fix k general points p_i on Q and $l \geq 0$ general pairs of points q_j, q'_j with $k + 3l = r$, $r = 2d_1 + 2d_2 + 1$. Let $E^l((d_1, d_2), g)$ be the number of hyperelliptic curves on Q of genus g and bi-degree (d_1, d_2) passing through all the points and satisfying also the condition that q_i is hyperelliptically conjugate to q'_i for all i .*

Theorem 4.12. *With $\beta = (d_2, d_1, d_1 + d_2 - g - 1)$ and r, l as above:*

$$\langle T_{13}^l \cdot T_4^{r-3l} \rangle_\beta = \sum_{h \geq g} \binom{2h+2}{h-g} E^l((d_1, d_2), h) \quad (7)$$

Proof. We write $\beta = (a, b, c)$ where $a = d_2, b = d_1, c = d_1 + d_2 - g - 1$. Consider the case $l = 0$ and fix r general points p_1, \dots, p_r . Then the invariant $\langle T_4^r \rangle_\beta$ is given by the degree $\deg(ev^{-1}(\prod \Gamma(p_i)))$, where by Theorem 4.6 the scheme $ev^{-1}(\prod \Gamma(p_i))$ is supported in a finite set of points. By Corollary 4.8, the only contribution to the invariant comes from the component of the moduli space $\overline{M}_{0,r}(\mathbf{H}, \beta)$ corresponding to stable maps from comblike curves such that the irreducible (a, b, c_0) -component is incident to all the cycles $\Gamma(p_i)$. Hence the number of stable maps is equal to the number of possible irreducible curves of class (a, b, c_0) times the number of choices for the attachment points of the $(0, 0, 1)$ -curves. We have to choose $c - c_0$ points among the $2(a + b - c_0)$ ones in the intersection $(a, b, c_0) \cdot \Delta$. The formula then follows from the relationship between (a, b, c) and (d_1, d_2, g) .

If $l \geq 1$, then the curves have to meet also l general points of \mathbf{H} . Choosing representatives of the point class T_{13} outside $\Delta \cup \Sigma$, curves moving in excess dimension cannot satisfy this condition, so Theorem 4.6 applies also for $l \geq 1$. The same arguments conclude the proof. \square

Remark 4.13. We note that the sum (7) is finite, in fact the values of h are equal to $d_1 + d_2 - c_0 - 1$ with $c_0 \leq c$. By what we showed in section 3.5 if $l \geq 1$ then we can compute all the invariants $\langle T_{13}^l \cdot T_4^{r-3l} \rangle_\beta$. Therefore we

can invert the formula (7) to get the numbers $E^l((d_1, d_2), h)$.

The numbers $E^0((d_1, d_2), g)$ are zero for small values of d_1, d_2, g . In fact $E^0((d_1, d_2), g)$ is less then or equal to $S((d_1, d_1), g)$, the number of smooth curves of bi-degree (d_1, d_2) of genus g passing through r points. We know that $S((d_1, d_1), g)$ is zero if $d_1 d_2 - d_1 - d_2 - 1 < 0$, hence the first possibly nonzero GW invariants with $l = 0$ are $\langle T_4^{11} \rangle_{(3,2,2)}$ and $\langle T_4^{11} \rangle_{(2,3,2)}$.

4.4 Proof of Theorem 4.6

Lemma 4.14. *With notations as in Theorem 4.6, let C be an irreducible rational curve meeting all the cycles $\Gamma(p_i)$ and the orbit Σ_4 . Then it intersects $\Delta \cup \Sigma$ in points disjoint from all the $\Gamma(p_i)$.*

Proof. Let $r = 2a + 2b + 1$ and $M \subseteq \overline{M}_{0,r}(\mathbf{H}, (a, b, c))$ be the open subset of points $[C, \mu, x_j]_{j=1,\dots,r}$ such that $C \cong \mathbb{P}^1$, $\mu(C) \cap \Sigma_4 \neq \emptyset$. It is smooth of dimension $2r$. The map $M \rightarrow \overline{M}_{0,0}(\mathbf{H}, (a, b, c))$ which forgets the markings and stabilizes factors through:

$$M \xrightarrow{\pi_i} \overline{M}_{0,1}(\mathbf{H}, (a, b, c)) \longrightarrow \overline{M}_{0,0}(\mathbf{H}, (a, b, c))$$

where π_i is the map forgetting all the markings but x_i and stabilizing. It is surjective onto its image $\text{Im}(\pi_i) = \mathcal{U}_1$ which is the universal curve over the smooth locus \mathcal{U}_0 of $\overline{M}_{0,0}(\mathbf{H}, (a, b, c))$. Then $\pi_i : M \rightarrow \mathcal{U}_1$ is flat of relative dimension $r - 1$. The set $N = \{[C, \mu, x] : \mu(x) \in \Delta \cup \Sigma\}$ is a closed subset of \mathcal{U}_1 , as it is the inverse image $ev^{-1}(\Delta \cup \Sigma)$. Its complementary $\mathcal{U}_1 \setminus N$ is open and intersects all the 1-dimensional fibers of $\mathcal{U}_1 \rightarrow \mathcal{U}_0$, then it is dense. This implies that N is a proper closed subset, equivalently it has dimension lower than $r + 1$. Moreover the inverse image $M_i = \pi_i^{-1}(N)$ has dimension $\dim M_i < \dim M = 2r$ because also the restricted map $\pi_i : M_i \rightarrow N$ is flat of relative dimension $r - 1$. Let \tilde{M}_i be the resolution of singularities of M_i . It has the same dimension as M_i . Set $\Gamma = \prod_{i=1}^r \Gamma(p_i)$, for generic fixed points $p_1, \dots, p_r \in Q$ and consider the inverse image of Γ in \tilde{M}_i via the evaluation map, i.e. the composition:

$$ev_i : \tilde{M}_i \twoheadrightarrow M_i \xrightarrow{ev} \mathbf{H}^r$$

We apply the Position Lemma to ev_i with the group \mathcal{A}_0 acting on \mathbf{H} (by 1.8). By 1.7, $ev_i^{-1}(\Gamma)$ has pure dimension equal to $\dim M_i - \text{cod}(\Gamma \subseteq \mathbf{H}^r) < 0$, that is to say it is empty. In particular $ev^{-1}(\Gamma) \cap M_i = \emptyset$. \square

We are ready to give a proof of Theorem 4.6. We will use induction on the number of components of the source curve C and we will apply the Position Lemma with respect to the action of \mathcal{A}_0 on \mathbf{H} .

Proof. STEP 1. The subset $M = \{[C, \mu, x_j] : C \cong \mathbb{P}^1, \mu(C) \not\subseteq \Delta \cup \Sigma\}$ is open and smooth in $\overline{M}_{0,r}(\mathbf{H}, \beta)$ and we can consider the restriction of the evaluation map $ev : \overline{M}_{0,r}(\mathbf{H}, (a, b, c)) \rightarrow \mathbf{H}$ to it. Set $\Gamma = \prod_{i=1}^r \Gamma(p_i)$. By the Position Lemma, $\dim ev^{-1}(\Gamma) = 0$ since M is of the expected dimension $2r$.

STEP 2. Suppose that C is irreducible and $\mu(C) \subseteq \Delta$, then in Δ we have $\mu_*[C] = \tilde{\beta} = (a/2, b/2, c)$ by what we showed in §1.5. Let $a' = a/2, b' = b/2$. The image of $\mu(C)$ via the support map is a curve B of genus zero and bi-degree (a', b') on Q . The cycles $\Gamma(p_i)$ restricted to Δ have codimension 2 and the curve $\mu(C)$ is incident to all of them if and only if the image curve

B goes through all the points p_i . A rational curve on Q of bi-degree (a', b') passes through at most s generical points of Q , where:

$$2s = \dim Q + \int_{(a', b')} c_1(T_Q) - 3 + s \Rightarrow s = 2a' + 2b' - 1 = a + b - 1$$

We have $s < r = 2a + 2b + 1$, so the irreducible curves $\mu(C) \subseteq \Delta$ give no contribution to our calculations.

STEP 3. Now we analyse the contribution from irreducible rational curves C such that $\mu(C) \subseteq \Sigma$. Since Σ is the disjoint union $\tilde{W}_1 \sqcup \tilde{W}_2$ and $\mu(C)$ is irreducible, it is enough to consider the case $\mu(C) \subseteq \tilde{W}_1$. The pushforward class $\mu_*[C]$ in \tilde{W}_1 is $(a, b/2)$ with b even. Let φ_1^r be the map induced by the restricted blowup $\varphi_1 : \tilde{W}_1 \rightarrow W_1$. Then we have a composition map:

$$\overline{M}_{0,r}(\tilde{W}_1, (a, b/2)) \xrightarrow{ev} \tilde{W}_1^r \xrightarrow{\varphi_1^r} W_1^r \subseteq \mathbf{G}^r$$

If a curve of class $(a, b/2)$ intersects all the cycles $\Gamma(p_i)$ then its image via φ_1 is of class $(\varphi_1)_*(a, b/2) = b/2 \cdot [W_1] = b[\sigma_{2,1}]$ because W_1 is a quadric in \mathbf{G} , and it goes through all the points $l_1(p_i) \in \mathbf{G}$. Such a curve passes through at most s fixed points in \mathbf{G} , with s given by the formula:

$$4s = \dim \mathbf{G} + \int_{b[\sigma_{2,1}]} c_1(T_{\mathbf{G}}) - 3 + s \Rightarrow s = \frac{1 + 4b}{3}$$

Since $s < r$ we verify that irreducible curves mapped into Σ give no contribution to our computation.

Suppose that C is the union of k irreducible components and $\mu(C) \subseteq \Sigma$. Since \tilde{W}_1, \tilde{W}_2 are disjoint, if an irreducible component is mapped into \tilde{W}_i then all the components are actually mapped into the same divisor \tilde{W}_i , by connectedness. We can assume $\mu(C) \subseteq \tilde{W}_1$. The number k of components is bounded. In fact $\mu_*[C] = (a, b)$ in \mathbf{H} , with b even, then k is at most equal to $a + b/2$. This implies that $\mu(C)$ goes through at most $s = \frac{k+4b}{3} \leq \frac{2a+9b}{6}$ cycles. We get $s < r$ also in this case.

Lemma 4.14 concludes the proof of 1.-2.

STEP 4. Suppose C is reducible and $\mu(C) \subseteq \Delta \cup \Sigma$. In particular assume that C has k irreducible components C_i such that:

$$\begin{aligned} C_i &\subseteq \Delta & \text{for } 1 \leq i \leq k_1 \\ C_i &\subseteq \tilde{W}_1 & \text{for } k_1 + 1 \leq i \leq k_2 \\ C_i &\subseteq \tilde{W}_2 & \text{for } k_2 + 1 \leq i \leq k \end{aligned}$$

We fix the notations:

$$D_1 = \bigcup_{i=1}^{k_1} C_i \quad \text{is of class } (a_1, b_1, c_1)$$

$$D_2 = \bigcup_{i=k_1+1}^{k_2} C_i \quad \text{is of class } (a_2, b_2, c_2)$$

$$D_3 = \bigcup_{i=k_2+1}^k C_i \quad \text{is of class } (a_3, b_3, c_3)$$

The conditions $\sum a_j = a$, $\sum b_j = b$, $\sum c_j = c$ hold. The image curve $\mu(D_j)$ intersects r_j cycles. By the previous results we know that $r_j \leq 2a_j + 2b_j$ for all j then $r_1 + r_2 + r_3 \leq 2a + 2b < r$. The curve C does not intersects all the cycles $\Gamma(p_i)$.

STEP 5. Let R be the subset of $\overline{M}_{0,r}(\mathbf{H}, (a, b, c))$ parametrizing stable maps $[C, \mu, x_j]$ such that $C = \bigcup C_i$ and each C_i intersects Σ_4 , the dense orbit. It is a proper closed subset of the smooth locus, hence it has dimension lower

than $2r$. For generic points p_i the intersection $R \cap ev^{-1}(\Gamma)$ is empty. Finally we analyse the contribution from stable maps $\mu : C \rightarrow \mathbf{H}$ with rational reducible domain and such that there exists at least one component of C mapped into $\Delta \cup \Sigma$. We can write $C = C_0 \cup C_1$ with $C_0 \cap C_1 = \{p\}$ a point mapped in $\Delta \cup \Sigma$. Set $[\mu(C_i)] = (a_i, b_i, c_i)$, with $\sum a_i = a$, $\sum b_i = b$, $\sum c_i = c$. We suppose $(a_i, b_i) \neq (0, 0)$ for $i = 0, 1$ and let $\mu(C)$ be incident to all the cycles $\Gamma(p_i)$. We know that $\mu(C_i)$ intersects $r_i = 2a_i + 2b_i + 1 - k_i$ cycles, with $k_0 + k_1 \leq 1$. Assume $r_0 = 2a_0 + 2b_0 + 1$ and $r_1 = 2a_1 + 2b_1$. Theorem 4.6 applies to C_0 , by induction. It implies that $\mu(C_1)$ intersects $r_1 + 1$ cycles and a point of intersection is in $\Delta \cup \Sigma$ (the proof is exactly the same as in [Gr] Thm. 2.7). This is impossible. Then $a_1 = b_1 = 0$. \square

Remark 4.15. Note that because of smoothness in Corollary 4.7 hyperelliptic curves are enumerated with multiplicity one. This fact was not proven in [P].

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